International Journal of Applied Mathematics

Volume 38 No. 2 2025, 253–270
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v38i2.6

SEMIGROUPS IN WHICH THE RADICAL OF EVERY BI-INTERIOR IDEAL IS A SUBSEMIGROUP

Wichayaporn Jantanan [1], Suphawan Phalasak [1], Supattra Tiangtas [1] and Ronnason Chinram [2,§]

[1] Department of Mathematics, Faculty of Science
Buriram Rajabhat University
Muang, Buriram, 31000 THAILAND
e-mails: wichayaporn.jan@bru.ac.th (W. Jantanan)
620112210032@bru.ac.th (S. Phalasak)
620112210036@bru.ac.th (S. Tiangtas)

[2] Division of Computational Science, Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla, 90110 THAILAND
e-mail: ronnason.c@psu.ac.th ([§] corresponding author)

Received: 5 February 2025

Abstract

In this paper, we focus on studying the radical of bi-interior ideals of semigroups. We characterize when the radical of every bi-interior ideal is a subsemigroup, a right ideal, a left ideal, a quasi-ideal, an ideal, a bi-ideal, an interior ideal and a bi-interior ideal. Also, the radical of every subsemigroup, right ideal, left ideal, quasi-ideal, ideal, bi-ideal, interior ideal and bi-interior ideal is a bi-interior ideal.

Math. Subject Classification: 20M11, 20M12

Key Words and Phrases: semigroup, radical, subsemigroup, bi-interior ideal

1. Introduction and preliminaries

A semigroup is one of the algebraic structures which was widely studied. Ideal theory of semigroups plays an important role in studying. In [1], Bogdanovic and Ciric characterized semigroups in which the radical of every ideal is a subsemigroup and the radical of every bi-ideal is a subsemigroup. Later, Sanborisoot and Changphas characterized semigroups and ordered semigroups in which the radical of every quasi-ideal is a subsemigroup in [5] and [6], respectively. The notion of bi-interior ideals was introduced by Rao in [8] which is a generalization of several standard ideals, for example, left ideals, ring ideals, ideals, quasi-ideals, bi-ideals and interior ideals of semigroups. We characterize when the radical of every bi-interior ideal is a subsemigroup, a right ideal, a left ideal, a quasi-ideal, an ideal, a bi-ideal, an interior ideal and a bi-interior ideal. Also, the radical of every subsemigroup, right ideal, left ideal, quasi-ideal, ideal, bi-ideal, interior ideal and bi-interior ideal is a bi-interior ideal.

We give some definitions and results which will be used throughout this paper. Those can be found in [1, 2, 4, 5, 7, 8].

Definition 1.1. A *semigroup* is a set S together with a binary operation $\cdot : S \times S \to S$ that satisfies the associative property:

$$\text{for all } x, y, z \in S,\ (xy)z = x(yz).$$

Let S be a semigroup and A, B non-empty subsets of S. The set product AB of A and B is defined to be the set of all elements ab with a in A and b in B. That is,

$$AB = \{ab \mid a \in A, b \in B\}.$$

For $a \in S$, we write Ba for $B\{a\}$, and similarly for aB.

Definition 1.2. A non-empty subset T of a semigroup S is called a *subsemigroup* of S if for all $x, y \in T,\ xy \in T$.

For $a, b \in S$, the subsemigroup of a semigroup S generated by $\{a, b\}$ is denoted by $\langle a, b \rangle$.

Definition 1.3. A non-empty subset T of a semigroup S is called a *left (right) ideal* of S if $ST \subseteq T$ $(TS \subseteq T)$.

Definition 1.4. A non-empty subset T of a semigroup S is called a *two-sided ideal* (or an *ideal*) of S if it is both a left ideal and a right ideal of S.

Definition 1.5. A non-empty subset Q of a semigroup S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$.

Definition 1.6. A subsemigroup B of a semigroup S is called a *bi-ideal* of S if $BSB \subseteq B$.

Definition 1.7. A subsemigroup I of a semigroup S is called an *interior ideal* of S if $SIS \subseteq I$.

Definition 1.8. A non-empty subset B of a semigroup S is called a *bi-interior ideal* of S if B is a subsemigroup of S and $SBS \cap BSB \subseteq B$.

For a non-empty subset A of a semigroup S, $\sqrt{A}$ denotes the *radical* of S, that is

$$\sqrt{A} = \{a \in S \mid a^n \in A \text{ for some positive integer } n\}.$$

Let $\mathbb{N} = \{1, 2, 3, ...\}$ denote the set of all positive integers. Let a, b, c be any elements of a semigroup S with identity. Define

$$\begin{aligned}
a \mid b &\iff b = xay \text{ for some } x, y \in S;\\
a \mid_r b &\iff b = ax \text{ for some } x \in S;\\
a \mid_l b &\iff b = ya \text{ for some } y \in S;\\
ab \mid_b c &\iff c = axb \text{ for some } x \in S;\\
a \mid_t b &\iff a \mid_r b \wedge a \mid_l b;\\
a \to b &\iff a \mid b^n \text{ for some } n \in \mathbb{N}; \text{ and}\\
a \xrightarrow{h} b &\iff a \mid_h b^n \text{ for some } n \in \mathbb{N} \text{ where } h \text{ is } r, l, b \text{ or } t.
\end{aligned}$$

2. **Main results**

We begin this section with the following theorem characterizes when the radical of every bi-interior ideal of a semigroup is a subsemigroup.

THEOREM 2.1. *Let S be a semigroup with identity. Then the radical of every bi-interior ideal of S is a subsemigroup of S if and only if*

$$\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}].$$

P r o o f. Assume first that the radical of every bi-interior ideal of S is a subsemigroup of S. Let $a, b \in S$, and let $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Then B is a bi-interior ideal of S because

$$\begin{aligned}
BB &= (S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\})(S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\})\\
&\subseteq S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} = B
\end{aligned}$$

and

$$\begin{aligned}
SBS \cap BSB = {} & S(S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\})S \cap (S\{a^i, b^j\}S\\
& \cap \{a^i, b^j\}S\{a^i, b^j\})S(S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\})\\
\subseteq {} & S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} = B.
\end{aligned}$$

Since $a^i, b^j \in B$, we have that $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a subsemigroup of S, and so $ab \in \sqrt{B}$. Thus,

$$(ab)^n \in B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \text{ for some } n \in \mathbb{N}.$$

Conversely, assume that for all $a, b \in S$ and $i, j \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Let B be a bi-interior ideal of S. To show that $\sqrt{B}$ is a subsemigroup of S, let $a, b \in \sqrt{B}$.

Then $a^i, b^j \in \sqrt{B}$ for some $i, j \in \mathbb{N}$. By assumption, there exists $n \in \mathbb{N}$ such that

$$(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \subseteq SBS \cap BSB \subseteq B.$$

Thus, $(ab)^n \in B$ for some $n \in \mathbb{N}$. Hence, $ab \in \sqrt{B}$. This shows that $\sqrt{B}$ is a submigroup of S. $\square$

EXAMPLE 2.1. Let $S = \{a, b, c, d, e\}$ be a semigroup ([3]) with the multipication:

$\cdot$	a	b	c	d	e
a	e	b	a	d	e
b	b	b	b	b	b
c	a	b	c	d	e
d	d	b	d	d	d
e	e	b	e	d	e

The bi-interior ideal of S is $\{\{b\}, \{b, d\}, \{b, d, e\}, \{a, b, d, e\}, S\}$. Observe that $\sqrt{\{b\}} = \{b\}$, $\sqrt{\{b, d\}} = \{b, d\}$, $\sqrt{\{b, d, e\}} = \{a, b, d, e\}$, $\sqrt{\{a, b, d, e\}} = \{a, b, d, e\}$ and $\sqrt{S} = S$. Then the radical of every bi-interior ideal of S is a subsemigroup of S.

In general, the radical of bi-interior ideals of a semigroup with identity need not be a subsemigroup, as the following example:

EXAMPLE 2.2. Let $S = \{a, b, c, d, f, 1\}$ be a semigroup ([5]) with the multipication:

$\cdot$	a	b	c	d	f	1
a	a	a	a	a	a	a
b	a	b	a	d	a	b
c	a	f	c	c	f	c
d	a	b	d	d	b	d
f	a	f	a	c	a	f
1	a	b	c	d	f	1

The bi-interior ideal of S is $\{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, f\}, \{a, b, d\}, \{a, c, d\}, \{a, b, f\}, \{a, c, f\}, \{a, b, c, d, f\}, S\}$. We have that the radical $\sqrt{\{a, c, d\}} = \{a, c, d, f\}$. But $\{a, c, d, f\}$ is not a subsemigroup of S. This shows that the radical of bi-interior ideals of a semigroup S with identity need not be a subsemigroup.

THEOREM 2.2. *Let S be a semigroup with identity. The radical of every bi-interior ideal of S is a right ideal of S if and only if*

$$\forall a, b \in S\ [a \mid_r b \Rightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [b^n \in Sa^kS \cap a^kSa^k]].$$

P r o o f. Assume that the radical of every bi-interior ideal of S is a right ideal of S. Let $a, b \in S$ and $k \in \mathbb{N}$ such that $a \mid_r b$. Then $b = ax$ for some $x \in S$. Setting $B = Sa^kS \cap a^kSa^k$. Then B is a bi-interior ideal of S and $a \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a right ideal of S. From $b = ax$, it follows that $b = ax \in \sqrt{B}S \subseteq \sqrt{B}$. Thus, $b \in \sqrt{B}$ and so $b^n \in B = Sa^kS \cap a^kSa^k$ for some $n \in \mathbb{N}$.

Conversely, assume that for all $a, b \in S$,

$$a \mid_r b \Rightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [b^n \in Sa^kS \cap a^kSa^k].$$

Let B be a bi-interior ideal of S. To show that $\sqrt{B}S \subseteq \sqrt{B}$, let $x \in \sqrt{B}S$. Then $x = ay$ for some $a \in \sqrt{B}$ and $y \in S$. Since $a \in \sqrt{B}$, then there exists $k \in \mathbb{N}$ such that $a^k \in B$. By assumption, there exists $n \in \mathbb{N}$ such that $x^n \in Sa^kS \cap a^kSa^k$. Thus,

$$x^n \in Sa^kS \cap a^kSa^k \subseteq SBS \cap BSB \subseteq B$$

for some $n \in \mathbb{N}$. Hence, $x^n \in B$ for some $n \in \mathbb{N}$, and so $x \in \sqrt{B}$. Therefore, $\sqrt{B}$ is a right ideal of S. □

Dually, we have the following theorem.

THEOREM 2.3. *Let S be a semigroup with identity. The radical of every bi-interior of S is a left ideal of S if and only if*

$$\forall a, b \in S\ [a \mid_l b \Rightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [b^n \in Sa^kS \cap a^kSa^k]].$$

THEOREM 2.4. *Let S be a semigroup with identity. The following conditions are equivalent:*

(1) *The radical of every bi-interior ideal of S is a quasi-ideal of S.*
(2) $\forall a, b, c \in S\ [a \mid_r c \wedge b \mid_l c \Rightarrow \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [c^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}]]$.

P r o o f. (1) $\Rightarrow$ (2) : Assume that (1) holds. Let $a, b, c \in S$ such that $a \mid_r c$ and $b \mid_l c$. Then $c = ay$ and $c = zb$ for some $y, z \in S$. Let $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Then B is a bi-interior ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a quasi-ideal of S. Since $c = ay$

and $c = zb$, it follows that

$$c \in \sqrt{B}S \cap S\sqrt{B} \subseteq \sqrt{B}.$$

Thus, $c \in \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that

$$c^n \in B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}.$$

(2) $\Rightarrow$ (1) : Assume that (2) holds. Let B be a bi-interior ideal of S. To show that $\sqrt{B}S \cap S\sqrt{B} \subseteq \sqrt{B}$., we let $x \in \sqrt{B}S \cap S\sqrt{B}$. Then $x = ay$ and $x = zb$ for some $y, z \in S$ and $a, b \in \sqrt{B}$. Since $a, b \in \sqrt{B}$, then $a^i, b^j \in B$ for some $i, j \in \mathbb{N}$. By assumption, there exists $n \in \mathbb{N}$ such that

$$x^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \subseteq SBS \cap BSB \subseteq B.$$

Thus, $x \in \sqrt{B}$. Hence, $\sqrt{B}$ is a quasi-ideal of S. □

THEOREM 2.5. *Let S be a semigroup with identity. The following conditions are equivalent:*

(1) *The radical of every bi-interior ideal of S is an ideal of S.*
(2) $\forall a, b \in S\ \forall k \in \mathbb{N}\ \exists m, n \in \mathbb{N}\ [(ab)^m, (ba)^n \in Sa^kS \cap a^kSa^k]$.

P r o o f. (1) $\Rightarrow$ (2) : Assume that (1) holds. Let $a, b \in S$ and let $k \in \mathbb{N}$. We set $B = Sa^kS \cap a^kSa^k$. Then B is a bi-interior ideal of S and $a \in \sqrt{B}$. By assumption, $\sqrt{B}$ is an ideal of S. So, we have that $ab \in \sqrt{B}S \subseteq \sqrt{B}$ and $ba \in S\sqrt{B} \subseteq \sqrt{B}$. Thus, $ab, ba \in \sqrt{B}$. Hence, there exist $m, n \in \mathbb{N}$ such that

$$(ab)^m, (ba)^n \in B = Sa^kS \cap a^kSa^k.$$

Conversely, assume that for all $a, b \in S$ and $k \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ such that $(ab)^m, (ba)^n \in Sa^kS \cap a^kSa^k$. Let B be a bi-interior ideal of S. To show that $\sqrt{B}$ is an ideal of S, let $a \in \sqrt{B}$ and $b \in S$. Since $a \in \sqrt{B}$, then there exist $k \in \mathbb{N}$ such that $a^k \in B$. By assumption, there exist $m, n \in \mathbb{N}$ such that

$$(ab)^m, (ba)^n \in Sa^kS \cap a^kSa^k \subseteq SBS \cap BSB \subseteq B.$$

Hence, $(ab)^m, (ba)^n \in B$ for some $m, n \in \mathbb{N}$, and so $ab, ba \in \sqrt{B}$. Therefore, $\sqrt{B}$ is an ideal of S. □

THEOREM 2.6. *Let S be a semigroup with identity. The radical of every bi-interior ideal of S is a bi-ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}]$;

(2) $\forall a, b, c \in S\ [ab \mid_b c \Rightarrow \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [c^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}]]$.

P r o o f. Assume that the radical of every bi-interior ideal of S is a bi-ideal of S. To show that (1) holds, we let $a, b \in S$, and $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Then B is a bi-interior ideal of S such that $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-ideal of S. So, we obtain that $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $(ab)^n \in B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Next, to show that (2) holds, we will let $a, b, c \in S$ such that $ab \mid_b c$. Then $c = axb$ for some $x \in S$. Let $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Then B is a bi-interior ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-ideal of S. From $c = axb$, it follows that

$$c = axb \in \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}.$$

Thus, $c \in \sqrt{B}$. Hence, $c^n \in B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let B be a bi-interior ideal of S. To show that $\sqrt{B}$ is a bi-ideal of S, we let $a, b \in \sqrt{B}$. Then $a^i \in B$ and $b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \subseteq SBS \cap BSB \subseteq B$. Hence, $ab \in \sqrt{B}$, and so $\sqrt{B}$ is a subsemigroup of S. Next, to show that $\sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$, let $x \in \sqrt{B}S\sqrt{B}$. Then $x = ayb$ for some $a, b \in \sqrt{B}$ and $y \in S$. Since $a, b \in \sqrt{B}$, then $a^i \in B$ and $b^j \in B$ for some $i, j \in \mathbb{N}$. By (2), there exists $n \in \mathbb{N}$ such that

$$x^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \subseteq SBS \cap BSB \subseteq B.$$

Thus, $x \in \sqrt{B}$. This shows that $\sqrt{B}$ is a bi-ideal of S. □

THEOREM 2.7. *Let S be a semigroup with identity. The radical of every bi-interior ideal of S is an interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}]$;

(2) $\forall a, b \in S\ [a \mid b \Rightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [b^n \in Sa^kS \cap a^kSa^k]]$.

P r o o f. Assume that the radical of every bi-interior of S is an interior ideal of S. To show that (1) holds, we let $a, b \in S$, and $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Then B is a bi-interior ideal of S and $a,b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is an interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $(ab)^n \in B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Next to show that (2) holds, we let $a, b \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Let $k \in \mathbb{N}$. We set $B = Sa^kS \cap a^kSa^k$. Then B is a bi-interior ideal of S. By assumption, $\sqrt{B}$ is an interior ideal of S and $a \in \sqrt{B}$. Since $b = xay$, we obtain that $b = xay \in S\sqrt{B}S \subseteq \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $b^n \in B = Sa^kS \cap a^kSa^k$.

Conversely, assume that (1) and (2) hold. Let B be a bi-interior ideal of S. We will show that $\sqrt{B}$ is an interior ideal of S. First, let $a, b \in \sqrt{B}$. Then $a^i \in B$ and $b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that

$$(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \subseteq SBS \cap BSB \subseteq B.$$

Thus, $(ab)^n \in B$ for some $n \in \mathbb{N}$. Hence, $ab \in \sqrt{B}$ and so $\sqrt{B}$ is a subsemigroup of S. Next, let $x \in S\sqrt{B}S$. Then $x = yaz$ for some $y, z \in S$ and $a \in \sqrt{B}$. Since $a \in \sqrt{B}$, then $a^k \in B$ for some $k \in \mathbb{N}$. By (2), there exists $n \in \mathbb{N}$ such that

$$x^n \in Sa^kS \cap a^kSa^k \subseteq SBS \cap BSB \subseteq B.$$

Thus, $x \in \sqrt{B}$. This shows that $\sqrt{B}$ is an interior ideal of S. □

THEOREM 2.8. *Let S be a semigroup with identity. The radical of every bi-interior ideal of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i.b^j\}]$;

(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \ [d^n \in S\{a^i, b^j, c^k\}S \cap \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\}]]$.

P r o o f. Assume that the radical of every bi-interior ideal of S is a bi-interior ideal of S. To show that (1) holds, we let $a, b \in S$ and $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Then B is a bi-interior ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $(ab)^n \in B = S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\}$. Next, to show that (2) holds. Let $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x, y, z \in S$. Let $i, j, k \in \mathbb{N}$. We set $B = S\{a^i, b^j, c^k\}S \cap \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\}$. Then B is a bi-interior ideal of S and $a, b, c \in \sqrt{B}$.

By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, it follow that
$$d \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}.$$
This implies that $d^n \in B = S\{a^i, b^j, c^k\}S \cap \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\}$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let B be a bi-interior ideal of S. We will show that $\sqrt{B}$ is a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$. Then $a^i, b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that
$$(ab)^n \in S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \subseteq SBS \cap BSB \subseteq B.$$
Thus, $ab \in \sqrt{B}$. Hence, $\sqrt{B}$ is subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$. Let $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. Then $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c \in \sqrt{B}$. Since $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (2), there exists $n \in \mathbb{N}$ such that
$$x^n \in S\{a^i, b^j, c^k\}S \cap \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\} \subseteq SBS \cap BSB \subseteq B.$$
Thus, $x \in \sqrt{B}$. This shows that $\sqrt{B}$ is a bi-interior ideal of S. $\square$

THEOREM 2.9. *Let S be a semigroup with identity. The radical of every right ideal of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in \{a^i, b^j\}S]$;

(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [d^n \in \{a^i, b^j, c^k\}S]]$.

P r o o f. Assume that the radical of every right ideal of S is a bi-interior ideal of S. To show that (1) holds, we let $a, b \in S$ and $i, j \in \mathbb{N}$. We set $B = \{a^i, b^j\}S$. Then B is a right ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $(ab)^n \in B = \{a^i, b^j\}S$. Next, to show that (2) holds. Let $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x, y, z \in S$. Let $i, j, k \in \mathbb{N}$. We set $B = \{a^i, b^j, c^k\}S$. Then B is a right ideal of S and $a, b, c \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, it follows that $d \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $d^n \in B = \{a^i, b^j\}S$.

Conversely, assume that (1) and (2) hold. Let B be a right ideal of S. We will show that $\sqrt{B}$ is a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$.

Then $a^i, b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in \{a^i, b^j\}S \subseteq BS \subseteq B$. Thus, $ab \in \sqrt{B}$. Hence, $\sqrt{B}$ is a subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$, we let $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. Then $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c \in \sqrt{B}$. Since $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in \{a^i, b^j, c^k\}S \subseteq BS \subseteq B$. Thus, $x \in \sqrt{B}$. Hence, $\sqrt{B}$ is a bi-interior ideal of S. □

Dually, we have the following theorem.

THEOREM 2.10. *Let S be a semigroup with identity. The radical of every left ideal of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in S\{a^i, b^j\}]$;
(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [d^n \in S\{a^i, b^j, c^k\}]]$.

THEOREM 2.11. *Let S be a semigroup with identity. The radical of every quasi-ideal of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}]$;
(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [d^n \in \{a^i, b^j, c^k\}S \cap S\{a^i, b^j, c^k\}]]$.

Proof. Assume that that radical of every quasi-ideal of S is a bi-interior of S. To show that (1) holds, let $a, b \in S$ and $i, j \in \mathbb{N}$. We set $B = \{a^i, b^j\}S \cap S\{a^i, b^j\}$. Then B is a quasi-ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq B$. Hence, $(ab)^n \in B = \{a^i, b^j\}S \cap S\{a^i, b^j\}$ for some $n \in \mathbb{N}$. Next, to show that (2) holds, let $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x, y, z \in S$. Let $i, j, k \in \mathbb{N}$. Put $B = \{a^i, b^j, c^k\}S \cap S\{a^i, b^j, c^k\}$. Then B is a quasi-ideal of S and $a, b, c \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, it follows that $d \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $d^n \in B = \{a^i, b^j, c^k\}S \cap S\{a^i, b^j, c^k\}$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let B be a quasi-ideal of S. We will show that $\sqrt{B}$ is a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$. Then $a^i, b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\} \subseteq BS \cap SB \subseteq B$. Thus, $ab \in \sqrt{B}$, and so $\sqrt{B}$ is a subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$, we consider $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. Then $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c \in \sqrt{B}$. Since $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (2), there exists $n \in \mathbb{N}$ such that

$$x^n \in \{a^i, b^j, c^k\}S \cap S\{a^i, b^j, c^k\} \subseteq BS \cap SB \subseteq B.$$

This shows that $\sqrt{B}$ is a bi-interior ideal of S. □

THEOREM 2.12. *Let S be a semigroup with identity. The radical of every ideal of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in S\{a^i, b^j\}S]$;

(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [d^n \in S\{a^i, b^j\}S]]$.

P r o o f. Assume that the radical of every ideal of S is a bi-interior ideal of S. To show that (1) holds, let $a, b \in S$ and $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S$. Then B is an ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, $(ab)^n \in B = S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$. Next, to show that (2) holds, we let $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x, y, z \in S$. Let $i, j, k \in \mathbb{N}$. We set $B = S\{a^i, b^j, c^k\}S$. Then B is an ideal of S and $a, b, c \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, it follows that $d \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$. Hence, $d^n \in B = S\{a^i, b^j, c^k\}S$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let B be an ideal of S. We will show that $\sqrt{B}$ is a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$. Then $a^i, b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^i, b^j\}S \subseteq SBS \subseteq BS \subseteq B$. Thus, $ab \in \sqrt{B}$. Hence, $\sqrt{B}$ is a subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$, we will let $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. Then $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c \in \sqrt{B}$. Since $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in S\{a^i, b^j, c^k\}S \subseteq SBS \subseteq BS \subseteq B$. Thus, $x \in \sqrt{B}$. Hence, $\sqrt{B}$ is a bi-interior ideal of S. □

THEOREM 2.13. *Let S be a semigroup with identity. The radical of every bi-ideal of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in \{a^i, b^j\}S\{a^i, b^j\}]$;

(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [d^n \in \{a^i, b^j\}S\{a^i, b^j\}]]$.

P r o o f. Assume that the radical of every bi-ideal of S is a bi-interior ideal of S. To show that (1) holds, we let $a, b \in S$ and let $i, j \in \mathbb{N}$. We set $B = \{a^i, b^j\}S\{a^i, b^j\}$. Then B is a bi-ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, $(ab)^n \in B = \{a^i, b^j\}S\{a^i, b^j\}$ for some $n \in \mathbb{N}$. Next, to show that (2) holds, we consider $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x, y, z \in S$. Let $i, j, k \in \mathbb{N}$. We set $B = \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\}$. Then B is a bi-ideal of S and $a, b, c \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, we obtain $d \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$. Hence, $d^n \in B = \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\}$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let B be a bi-ideal of S. We will show that $\sqrt{B}$ is a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$. Then $a^i, b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in \{a^i, b^j\}S\{a^i, b^j\} \subseteq BSB \subseteq B$. Thus, $ab \in \sqrt{B}$, and so $\sqrt{B}$ is a subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$, we let $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. We have $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c\sqrt{B}$. Since $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (2), there exists $n \in \mathbb{N}$ such that

$$x^n \in \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\} \subseteq BSB \subseteq B.$$

Thus, $x \in \sqrt{B}$. Hence, $\sqrt{B}$ is a bi-interior ideal of S. □

THEOREM 2.14. *Let S be a semigroup with identity. The radical of every interior ideal of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in S\{a^i, b^j\}S]$;

(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [d^n \in S\{a^i, b^j, c^k\}S]]$.

P r o o f. Assume that the radical of every interior ideal of S is a bi-interior ideal of S. To show that (1) holds, we let $a, b \in S$ and $i, j \in \mathbb{N}$. We set $B = S\{a^i, b^j\}S$. Then B is an interior ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, $(ab)^n \in B = S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$. Next, to show that (2) holds, let $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and

$d = bzc$ for some $x, y, z \in S$. Let $i, j, k \in \mathbb{N}$. We set $B = S\{a^i, b^j, c^k\}S$. Then B is an interior ideal of S and $a, b, c \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, thus $d \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$. Hence, $d^n \in B = S\{a^i, b^j, c^k\}S$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let B be an interior ideal of S. We will show that $\sqrt{B}$ is a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$. Then $a^i, b^j \in \sqrt{B}$ for some $i, j, k \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^i, b^j\}S \subseteq SBS \subseteq B$. Thus, $ab \in \sqrt{B}$, and so $\sqrt{B}$ is a subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$, we will consider $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. Then $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c \in \sqrt{B}$. Since $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in S\{a^i, b^j, c^k\}S \subseteq SBS \subseteq B$. Thus, $x \in \sqrt{B}$. Hence, $\sqrt{B}$ is a bi-interior ideal of S. $\square$

THEOREM 2.15. *Let S be a semigroup with identity. The radical of every subsemigroup of S is a bi-interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in \langle a^i, b^j \rangle]$;

(2) $\forall a, b, c, d \in S\ [a \mid d \wedge bc \mid_b d \Rightarrow \forall i, j, k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [d^n \in \langle a^i, b^j, c^k \rangle]]$.

P r o o f. Assume that the radical of every subsemigroup of S is a bi-interior ideal of S. To show that (1) holds, let $a, b \in S$ and $i, j \in \mathbb{N}$. We set $B = \langle a^i, b^j \rangle$. Then B is a subsemigroup of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Thus, $ab \in \sqrt{B}\sqrt{B} \subseteq \sqrt{B}$. Hence, $(ab)^n \in B = \langle a^i, b^j \rangle$ for some $n \in \mathbb{N}$. Next, to show that (2) holds, we let $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x, y, z \in S$. Let $i, j, k \in \mathbb{N}$. We set $B = \langle a^i, b^j, c^k \rangle$. Then B is a subsemigroup of S and $a, b, c \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, thus $d \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$. Hence, there exists $n \in \mathbb{N}$ such that $d^n \in B = \langle a^i, b^j, c^k \rangle$.

Conversely, assume that (1) and (2) hold. Let B be a subsemigroup of S. We will show that $\sqrt{B}$ is a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$. Then $a^i, b^j \in B$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in \langle a^i, b^j \rangle$. Since $\langle a^i, b^j \rangle \subseteq B$, it follows that $(ab)^n \in \langle a^i, b^j \rangle \subseteq B$ for some $n \in \mathbb{N}$. Thus, $ab \in \sqrt{B}$ and so $\sqrt{B}$ is a subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$ let $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$.

Then $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c \in \sqrt{B}$. Since $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in \langle a^i, b^j, c^k \rangle$. Since $\langle a^i, b^j, c^k \rangle \subseteq B$, it follows that $x^n \in B$ for some $n \in \mathbb{N}$. Thus, $x \in \sqrt{B}$. This show that $\sqrt{B}$ is a bi-interior ideal of S. □

Finally, we have the following result.

THEOREM 2.16. *Let S be a semigroup with identity. The following conditions are equivalent:*

(1) *The radical of every bi-interior ideal of S is a bi-interior ideal of S.*

(2) *For any $a, b, c \in S$, $\sqrt{S\{a,b\}S \cap \{a,b\}S\{a,b\}}$ and $\sqrt{S\{a,b,c\}S \cap \{a,b,c\}S\{a,b,c\}}$ are bi-interior ideals of S.*

(3) *For any $a, b, c, d \in S$,*

(3.1) *there exists $n \in \mathbb{N}$ such that*

$$(ab)^n \in S\{a^2, b^2\}S \cap \{a^2, b^2\}S\{a^2, b^2\};$$

(3.2) *if $a \mid d$ and $bc \mid_b d$, then there exists $n \in \mathbb{N}$ such that*

$$d^n \in S\{a^2, b^2, c^2\}S \cap \{a^2, b^2, c^2\}S\{a^2, b^2, c^2\}.$$

(4) *For any $a, b, c, d \in S$ and $k \in \mathbb{N}$,*

(4.1) *there exists $n \in \mathbb{N}$ such that*

$$(ab)^n \in S\{a^k, b^k\}S \cap \{a^k, b^k\}S\{a^k, b^k\};$$

(4.2) *if $a \mid d$ and $bc \mid_b d$, then there exists $n \in \mathbb{N}$ such that*

$$d^n \in S\{a^k, b^k, c^k\}S \cap \{a^k, b^k, c^k\}S\{a^k, b^k, c^k\}.$$

P r o o f. (1) $\Rightarrow$ (2) : Assume that the radical of every bi-interior ideal of S is a bi-interior ideal of S and let $a, b, c \in S$. Since $S\{a,b\}S \cap \{a,b\}S\{a,b\}$ is a bi-interior of S and by assumption, $\sqrt{S\{a,b\}S \cap \{a,b\}S\{a,b\}}$ is a bi-interior ideal of S. Similarly, we have $S\{a,b,c\}S \cap \{a,b,c\}S\{a,b,c\}$ is a bi-interior ideal of S, and so $\sqrt{S\{a,b,c\}S \cap \{a,b,c\}S\{a,b,c\}}$ is a bi-interior ideal of S.

(2) $\Rightarrow$ (3) : First, to show that (3.1) holds, we let $a, b \in S$. Clearly, $a, b \in \sqrt{S\{a^2,b^2\}S \cap \{a^2,b^2\}S\{a^2,b^2\}}$. By (2), $\sqrt{S\{a^2,b^2\}S \cap \{a^2,b^2\}S\{a^2,b^2\}}$

is a bi-interior ideal of S. Thus,

$$\begin{aligned} ab &\in (\sqrt{S\{a^2,b^2\}S\cap\{a^2,b^2\}S\{a^2,b^2\}})(\sqrt{S\{a^2,b^2\}S\cap\{a^2,b^2\}S\{a^2,b^2\}}) \\ &\subseteq \sqrt{S\{a^2,b^2\}S\cap\{a^2,b^2\}S\{a^2,b^2\}}. \end{aligned}$$

Hence, $d^n \in S\{a^2,b^2\}S\cap\{a^2,b^2\}S\{a^2,b^2\}$ for some $n\in\mathbb{N}$. Thus, (3.1) holds. Next, to show that (3.2) holds, we consider $a,b,c,d\in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x,y,z\in S$. Clearly, $a,b,c \in \sqrt{S\{a^2,b^2,c^2\}S\cap\{a^2,b^2,c^2\}S\{a^2,b^2,c^2\}}$. By (2), $\sqrt{S\{a^2,b^2,c^2\}S\cap\{a^2,b^2,c^2\}S\{a^2,b^2,c^2\}}$ is a bi-interior ideal of S. Since $d = xay$ and $d = bzc$, it follows that

$$\begin{aligned} d &\in S(\sqrt{S\{a^2,b^2,c^2\}S\cap\{a^2,b^2,c^2\}S\{a^2,b^2,c^2\}})S \\ &\quad \cap (\sqrt{S\{a^2,b^2,c^2\}S\cap\{a^2,b^2,c^2\}S\{a^2,b^2,c^2\}})S \\ &\quad (\sqrt{S\{a^2,b^2,c^2\}S\cap\{a^2,b^2,c^2\}S\{a^2,b^2,c^2\}}) \\ &\subseteq \sqrt{S\{a^2,b^2,c^2\}S\cap\{a^2,b^2,c^2\}S\{a^2,b^2,c^2\}}. \end{aligned}$$

Hence, $d^n \in S\{a^2,b^2,c^2\}S\cap\{a^2,b^2,c^2\}S\{a^2,b^2,c^2\}$ for some $n\in\mathbb{N}$. This show that (3.2) holds.

(3) $\Rightarrow$ (4) : First, to show that (4.1) holds, let $a,b\in S$. By (3.1), $(ab)^n \in S\{a^2,b^2\}S\cap\{a^2,b^2\}S\{a^2,b^2\}$ for some $n\in\mathbb{N}$. It follows that

$$\begin{aligned} (ab)^n &\in S\{a^2,b^2\}S\cap\{a^2,b^2\}S\{a^2,b^2\} \\ &\subseteq S\{a,b\}S\cap\{a,b\}S\{a,b\}. \end{aligned}$$

Hence, there exists $n\in\mathbb{N}$ such that $(ab)^n\in S\{a,b\}S\cap\{a,b\}S\{a,b\}$. Suppose that there exists $m\in\mathbb{N}$ where $k\in\mathbb{N}$ such that $(ab)^m \in S\{a^k,b^k\}S\cap \{a^k,b^k\}S\{a^k,b^k\}$. By (3.1), there exists $l\in\mathbb{N}$ such that $((ab)^m)^l \in S\{a^{2k},b^{2k}\}$
$S\cap\{a^{2k},b^{2k}\}S\{a^{2k},b^{2k}\}$. Consider

$$\begin{aligned} ((ab)^m)^l &\in S\{a^{2k},b^{2k}\}S\cap\{a^{2k},b^{2k}\}S\{a^{2k},b^{2k}\} \\ &= S\{a^{k+1}a^{k-1},b^{k+1}b^{k-1}\}S \\ &\quad \cap\{a^{k+1}a^{k-1},b^{k+1}b^{k-1}\}S\{a^{k-1}a^{k+1},b^{k-1}b^{k+1}\} \\ &\subseteq S\{a^{k+1},b^{k+1}\}S\cap\{a^{k+1},b^{k+1}\}S\{a^{k+1},b^{k+1}\}. \end{aligned}$$

Hence,

$$(ab)^{ml} = ((ab)^m)^l \in S\{a^{k+1},b^{k+1}\}S\cap\{a^{k+1},b^{k+1}\}S\{a^{k+1},b^{k+1}\}.$$

Thus, (4.1) holds. Next, to show that (4.2) holds. Let $a, b, c, d \in S$ such that $a \mid d$ and $bc \mid_b d$. Then $d = xay$ and $d = bzc$ for some $x, y, z \in S$. By (3.2), there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} d^n &\in S\{a^2, b^2, c^2\}S \cap \{a^2, b^2, c^2\}S\{a^2, b^2, c^2\} \\ &\subseteq S\{a, b, c\}S \cap \{a, b, c\}S\{a, b, c\}. \end{aligned}$$

Hence, there exists $n \in \mathbb{N}$ such that $d^n \in S\{a, b, c\}S \cap \{a, b, c\}S\{a, b, c\}$. Suppose that there exists $m \in \mathbb{N}$ where $k \in \mathbb{N}$ such that $d^m \in S\{a^k, b^k, c^k\}$

$S \cap \{a^k, b^k, c^k\}S\{a^k, b^k, c^k\}$. By (3.2), there exists $l \in \mathbb{N}$ such that $(d^m)^l \in S\{a^{2k}, b^{2k}, c^{2k}\}S \cap \{a^{2k}, b^{2k}, c^{2k}\}S\{a^{2k}, b^{2k}, c^{2k}\}$. Consider

$$\begin{aligned} (d^m)^l &\in S\{a^{2k}, b^{2k}, c^{2k}\}S \cap \{a^{2k}, b^{2k}, c^{2k}\}S\{a^{2k}, b^{2k}, c^{2k}\} \\ &= S\{a^{k+1}a^{k-1}, b^{k+1}b^{k-1}, c^{k+1}c^{k-1}\}S \\ &\quad \cap \{a^{k+1}a^{k-1}, b^{k+1}b^{k-1}, c^{k+1}c^{k-1}\}S\{a^{k-1}a^{k+1}, b^{k-1}b^{k+1}, c^{k-1}c^{k+1}\} \\ &\subseteq S\{a^{k+1}, b^{k+1}, c^{k+1}\}S \cap \{a^{k+1}, b^{k+1}, c^{k+1}\}S\{a^{k+1}, b^{k+1}, c^{k+1}\}. \end{aligned}$$

Hence,

$$d^{ml} = (d^m)^l \in S\{a^{k+1}, b^{k+1}, c^{k+1}\}S \cap \{a^{k+1}, b^{k+1}, c^{k+1}\}S\{a^{k+1}, b^{k+1}, c^{k+1}\}.$$

Thus, (4.2) holds.

(4)⇒(1): Let B be a bi-interior ideal of S. First, let $a, b \in \sqrt{B}$. Then there exists $i, j \in \mathbb{N}$ such that $a^i, b^j \in B$. By (4.1), there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} (ab)^n &\in S\{a^{i+j}, b^{i+j}\}S \cap \{a^{i+j}, b^{i+j}\}S\{a^{i+j}, b^{i+j}\} \\ &= S\{a^i a^j, b^i b^j\}S \cap \{a^i a^j, b^j b^i\}S\{a^j a^i, b^i b^j\} \\ &\subseteq S\{a^i, b^j\}S \cap \{a^i, b^j\}S\{a^i, b^j\} \\ &\subseteq SBS \cap BSB \subseteq B. \end{aligned}$$

Thus, $ab \in \sqrt{B}$, and hence $\sqrt{B}$ is a subsemigroup of S. Next, to show that $S\sqrt{B}S \cap \sqrt{B}S\sqrt{B} \subseteq \sqrt{B}$. Let $x \in S\sqrt{B}S \cap \sqrt{B}S\sqrt{B}$. Then $x = way$ and $x = bzc$ for some $w, y, z \in S$ and $a, b, c \in \sqrt{B}$, then there exist $i, j, k \in \mathbb{N}$ such that $a^i, b^j, c^k \in B$. By (4.2), there exists $n \in \mathbb{N}$ such that $x^n \in S\{a^{i+j+k}, b^{i+j+k}, c^{i+j+k}\}S \cap \{a^{i+j+k}, b^{i+j+k}, c^{i+j+k}\}S\{a^{i+j+k}, b^{i+j+k}, c^{i+j+k}\}$. Consider

$$\begin{aligned} x^n &\in S\{a^{i+j+k}, b^{i+j+k}, c^{i+j+k}\}S \\ &\quad \cap \{a^{i+j+k}, b^{i+j+k}, c^{i+j+k}\}S\{a^{i+j+k}, b^{i+j+k}, c^{i+j+k}\} \\ &\subseteq S\{a^i, b^j, c^k\}S \cap \{a^i, b^j, c^k\}S\{a^i, b^j, c^k\} \subseteq SBS \cap BSB \subseteq B. \end{aligned}$$

Hence, $x \in \sqrt{B}$. Therefore, $\sqrt{B}$ is a bi-interior ideal of S. $\square$

References

[1] S. Bogdanovic and M. Ciric, Semigroups in which the radical of every ideal is a subsemigroup, *Zbornik Rad. Fil. Fak. Nis. Ser. Mat.*, **6** (1992), 129-135.

[2] R.A. Good and D.R. Hughes, Associated groups for a semigroup, *Bull. Amer. Math. Soc.*, **58** (1952), 624-625.

[3] N. Kehayopulu and M. Tsingelis, On left regular and intra-regular ordered semigroups, *Math. Slovaca*, **64** (2014), 1123-1134.

[4] S. Lajos, (m, k, n)-ideals in semigroups: notes on semigroups II, *Karl Marx Univ. Econ. Dept. Math. Budapest*, **1** (1976), 12-19.

[5] J. Sanborisoot and T. Changphas, Semigroups in which the radical of every quasi-ideal is a subsemigroup, *Quasigroups Related Syst.*, **28** (2020), 301-308.

[6] J. Sanborisoot and T. Changphas, Ordered semigroups in which the radical of every quasi-ideal is a subsemigroup, *Int. J. Math. Comput. Sci.*, **16** (2021), 1385-1396.

[7] O. Steinfeld, Uber die quasiideals von halbgruppen, *Publ. Math., Debrecen*, **4** (1956), 262-275.

[8] M.M.K. Rao, Bi-interior ideals of semigroups, *Discuss. Math. Gen. Algebra Appl.*, **38** (2018), 69-78.