International Journal of Applied Mathematics

Volume 38 No. 3 2025, 335–349
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v38i3.2

SOLVING THE NEGATIVE ORDER KORTEWEG-DE VRIES EQUATION WITH A SELF-CONSISTENT SOURCE CORRESPONDING TO MOVING EIGENVALUES

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Abstract

This study focuses on addressing the negative order Korteweg–de Vries (KdV) equation with a self-consistent source associated with dynamic eigenvalues, using the inverse scattering transform (IST). The primary goal is to establish the evolution of the scattering data for the spectral problem linked to the negative-order Korteweg–de Vries equation with a self-consistent source and moving eigenvalues. The derived relationships fully describe the evolution of the scattering data, enabling the application of the IST technique to solve the given problem.

Math. Subject Classification: 34L25, 35Q41, 35R30, 34M46

Key Words and Phrases: negative order Korteweg–de Vries equation, self-consistent source, inverse scattering transform, eigenvalue, eigenfunction, soliton solution

Received: February 19, 2025

1. **Introduction**

One of the key areas of research in mathematical physics involves the analysis of water waves. To describe their behavior, various integrable systems are used, often expressed as nonlinear partial differential equations such as the Korteweg–de Vries equation (KdV), Burgers equation, modified Korteweg–de Vries equation (mKdV), sine-Gordon equation and nonlinear Schrödinger equation (NLSE). Investigating solitons and these integrable systems extends beyond fluid mechanics, finding applications in fields such as nonlinear optics, classical and quantum field theory, and more.

In 1967, American researchers Gardner, Green, Kruskal, and Miura introduced the inverse scattering transform (IST) method to solve the Cauchy problem for the KdV equation, [1]. This method was initially applied to the Sturm–Liouville equation, as described in their seminal work. The KdV equation is given by:

$$u_t - 6uu_x + u_{xxx} = 0, \quad u = u(x,t).$$

Shortly thereafter, Lax generalized this powerful approach by formulating the Lax equation [2]:

$$L_t = [L, A],$$

where $[L, A] = LA - AL$ represents the commutator of the operators L and A. Here, L is the Sturm–Liouville operator:

$$Ly \equiv -y'' + u(x,t)y,$$

and A is a skew-symmetric operator acting in a Hilbert space. The IST method relies on reconstructing the potential of the Sturm–Liouville operator from scattering data [3], providing a framework to solve a wider class of integrable equations.

In 1987, V.K. Melnikov expanded the Lax equation to the form:

$$L_t = [L, A] + C,$$

where C consists of differential operators with coefficients derived from solutions to the spectral problem for L. Melnikov's work [4, 5] integrated the KdV equations with self-consistent sources using the IST method, particularly for rapidly decreasing functions. The term "self-consistent source" was first introduced in his studies and later applied in physical contexts, as seen in the work of J. Leon and A. Latifi [6].

Although much of the research focused on the positive-order KdV hierarchy, less attention was paid to the negative-order KdV hierarchy.

Verosky explored symmetries and negative powers of recursion operators [7], while Lou derived additional symmetries using invertible recursion operators, leading to the negative order KdV (NKdV) equation [8]:

$$u_t = 2vv_x, \quad v_{xx} + uv = 0 \quad \Longleftrightarrow \quad \left(\frac{v_{xx}}{v}\right)_t + 2vv_x = 0.$$

Subsequent studies investigated the Hamiltonian structures, Lax pairs, conservation laws, and soliton solutions of the NKdV equations [9, 10, 11].

Recent research has emphasized the integration of nonlinear evolution equations with self-consistent sources. Various methods have been developed, including the (G'/G) - expansion method [12, 13, 14, 15, 16], the Hirota direct method [17, 18], the inverse scattering transform [4, 5, 20], the inverse spectral problems [20, 21, 22] and the Darboux transformation [23].

In this paper, we apply the IST method to solve the NKdV equation with a self-consistent source associated with moving eigenvalues. The derived relationships fully describe the evolution of the scattering data, enabling the application of the IST method to the problem under consideration.

2. Statement of the problem

Consider the following NKdV equation with a self-consistent source that corresponds to moving eigenvalues

$$\begin{cases} u_t = 2vv_x + 2\sum_{m=1}^{N} \frac{\partial}{\partial x}(\varphi_m \psi_m), \\ v_{xx} = uv, \\ -\varphi_m'' + u\varphi_m = \lambda_m \varphi_m, \\ -\psi_m'' + u\psi_m = \lambda_m \psi_m, \quad m = 1, 2, ..., N, \quad t > 0, \ x \in R, \end{cases} \tag{1}$$

under the initial condition

$$u|_{t=0} = u_0(x), \quad x \in R, \tag{2}$$

where the initial function $u_0(x)$ has the following properties:

(1) $\int_{-\infty}^{\infty} (1 + |x|)|u_0(x)|dx < \infty$.
(2) The operator $L(y) \equiv -y'' + u_0(x)y = \lambda y, x \in R^1$ has exactly N number of negative eigenvalues $\lambda_1(0), \ \lambda_2(0), ..., \lambda_N(0)$.

In the problem considered, $\varphi_m(x,t)-$ is the eigenfunction of the operator $L(t) \equiv -\frac{d^2}{dx^2} + u(x,t)$ corresponding to the eigenvalue $\lambda_m(t) =$

$-\chi_m^2(t)$, and $\phi_m(x,t)-$ is linearly independent with the $\varphi_m(x,t)$ solution of the equation $L(t)y = \lambda_m y$, such that

$$W\left\{\varphi_m(x,t), \psi_m(x,t)\right\} = \omega_m(t), \quad m = 1, 2, ..., N. \tag{3}$$

Here $\omega_m(t),\ m = 1, 2, ..., N$ are given continuous functions that satisfy the condition for any non-negative t:

$$\int_o^t \omega_m(\tau)d\tau < -\lambda_m(0), \quad m = 1, 2, ..., N,$$

$$\omega_1(t) < \omega_2(t) < \omega_3(t) < ... < \omega_N(t). \tag{4}$$

It is assumed that the function $u(x,t)$ is sufficiently smooth, tends to zero under $x \to \pm\infty$ and $u(x,t)$, $v(x,t)$ tends to its limits rapidly enough when $x \to \pm\infty$ and satisfies the conditions:

$$\int_{-\infty}^{\infty} (1 + |x|) \left(|u(x,t)| + |u_t(x,t)|\right) dx < \infty,$$

$$\int_{-\infty}^{\infty} (1 + |x|) \left(|v_x(x,t)| + |v_{xx}(x,t)|\right) dx < \infty,$$

$$v^2(x,t) \to 1, \text{ as } |x| \to \pm\infty. \tag{5}$$

3. **Scattering problem**

In this section, the dependence of the function $u(x,t)$ on t is omitted. We consider the Sturm–Liouville equations on the axis [3]

$$Lg \equiv -g'' + u(x)g = k^2 g, \quad -\infty < x < \infty \tag{6}$$

with a real function $u(x)$ (potential) satisfying the "rapidly decreasing" condition

$$\int_{-\infty}^{\infty} (1 + |x|) |u(x)| \, dx < \infty. \tag{7}$$

In this part, we present the necessary information regarding the direct and inverse problems for equation (6).

Denote by $f(x,k)$ and $g(x,k)$ the Jost solutions of equation (6) with asymptotic:

$$\lim_{x\to-\infty} g(x,k)e^{ikx} = 1, \quad \lim_{x\to\infty} f(x,k)e^{-ikx} = 1, \quad Im\ k = 0. \tag{8}$$

Under condition (7), such solutions exist and are uniquely determined by asymptotics (8). For functions $\{f(x,k), f(x,-k)\}$ and $\{g(x,k), g(x,-k))\}$ are pairs of linearly independent solutions of equation (6), therefore:

$$f(x,k) = b(k)g(x,k) + a(k)g(x,-k),$$

$$g(x,k) = -b(-k)f(x,k) + a(k)f(x,-k), \quad Im\ k = 0. \tag{9}$$

By variable k, the Jost solutions $f(x,k)$ and $g(x,k)$ continue analytically to the upper half-plane $Im\ k > 0$. It is easy to see that the following equality is true:

$$a(k) = -\frac{1}{2ik} W\left\{f(x,k), g(x,k)\right\}, \tag{10}$$

where $W\{f,g\} = f(x,k)g'(x,k) - f'(x,k)g(x,k)$, moreover, for real k we have:

$$|a(k)|^2 = 1 + |b(k)|^2.$$

The function $a(k)$ is an analytic continuation to the half-plane $Im\ k > 0$ and has a finite number of simple zeros $k_n = i\chi_n, \quad n = 1, 2, ..., N$. Moreover, $\lambda_n = -\chi_n^2$ is an eigenvalue of the operator L. For $Im\ z > 0$ the function $a(z)$ is restored from its zeros $i\chi_n,\ n = 1, 2, ..., N$ and the function $r^+(k) = \frac{b(-k)}{a(k)}-$ defined on $Im\ k = 0$,

$$a(z) = \prod_{j=1}^{N} \frac{z - i\chi_j}{z + i\chi_j} \exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ln\left(1 - |r^+(k)|^2\right)}{k - z} dk\right\}.$$

According to (9), (10) and properties of the function $a(k)$,

$$g(x, i\chi_j) = B_j f(x, i\chi_j),\ j = 1, 2, ..., N. \tag{11}$$

For solutions $f(x,k), g(x,k)$ satisfying the representations

$$f(x,k) = e^{ikx} + \int_x^{\infty} A^+(x,z)e^{ikz}dz,$$

$$g(x,k) = e^{-ikx} + \int_{-\infty}^{x} A^-(x,z)e^{-ikz}dz, \tag{12}$$

the kernels $A^+(x,z), A^-(x,z)$ whose real functions are related to the potential $u(x)$ by the relations

$$u(x) = -2\frac{d}{dx}A^+(x,x), \quad u(x) = 2\frac{d}{dx}A^-(x,x). \tag{13}$$

The kernel $A^+(x,y)$ in representation (12) is a solution of the Gelfand–Levitan–Marchenko integral equation

$$\Omega^+(x+y)+A^+(x,y)+\int_x^\infty A^+(x,z)\Omega^+(z+y)dz=0 \quad (y>x), \quad (14)$$

where

$$\Omega^+(x)=-\sum_{j=1}^N \frac{iB_j}{\frac{da(z)}{dz}|_{z=i\chi_j}}e^{-\chi_j x}-\frac{1}{2\pi}\int_{-\infty}^\infty r^+(k)e^{ikx}dx,$$

and $a(z)-$ is the analytic continuation of the function $a(k)$ $(Im\ k=0)$ to the upper half-plane. Now the potential $u(x)$ is determined from equality (13).

The set $\{r^+(k), B_1, B_2, ..., B_N, \chi_1, \chi_2, ..., \chi_n\}$ is called the scatter data for problem (6)-(7).

Note that the function

$$h_n(x)=\frac{\frac{d}{dk}\left(g(x,k)-B_n f(x,k)\right)|_{k=i\chi_n}}{\dot{a}(i\chi_n)} \quad (15)$$

is a solution of the equation $L_0 y=-\chi_n^2 y$. According to (8) and (10) for $Im\ k>0$ we obtain the following asymptotics:

$$f(x,k)\sim a(k)e^{ikx} \ \text{ in } \ x\to-\infty,$$

$$g(x,k)\sim a(k)e^{-ikx} \ \text{ in } \ x\to\infty,$$

hence, these estimates and equality (15) imply that

$$h_n(x)\sim e^{\chi_n x} \ \text{ in } \ x\to\infty, \quad (16)$$

$$h_n(x)\sim -B_n e^{-\chi_n x} \ \text{ in } \ x\to-\infty. \quad (17)$$

Using (16) and (17), we obtain

$$W\left\{h_n(x), f(x,i\chi_n)\right\}=-2\chi_n,$$

$$W\left\{h_n(x), g(x,i\chi_n)\right\}=-2B_n\chi_n, \quad n=1,2,...,N. \quad (18)$$

4. **Evolution of scattering data**

In this section, we consider the system of equations:

$$\begin{cases} u_t = 2vv_x + G(x,t), \\ v_{xx} = uv, \end{cases} \tag{19}$$

where $G(x,t)$ - is a sufficiently smooth function for any non-negative t satisfying the conditions $G(x,t) = o(1)$ as $x \to \pm\infty$. Equations (19) are considered under the initial condition (2).

Solutions to the problem (19)-(2) are sought in the class of functions (5). For real k, we will look for the compatibility condition of the Lax pair for the system of equations (19) in the form

$$-g_{xx} + (u - k^2)g = 0, \tag{20}$$

$$g_t = -\frac{1}{2k^2}\left(v^2 g_x - vv_x g\right) + \left(-\frac{1}{2ik}\int_{-\infty}^{\infty} Gg\overline{g}dx + \frac{1}{2ik}\right)g + \frac{1}{2ik}\overline{g}\int_{-\infty}^{x} Gg^2 dx, \tag{21}$$

where $g = g(x,k,t), Im\ k = 0$. Passing in the last equality to the limit $x \to -\infty$ and by virtue of (5),(8),(9) we obtain:

$$a_t = -\frac{a}{2ik}\int_{-\infty}^{\infty} Gg\overline{g}dx - \frac{b}{2ik}\int_{-\infty}^{x} Gg^2 dx, \tag{22}$$

$$b_t = \frac{1}{ik}\overline{b} - \frac{\overline{b}}{2ik}\int_{-\infty}^{\infty} Gg\overline{g}dx - \frac{\overline{a}}{2ik}\int_{-\infty}^{x} Gg^2 dx,\ \ Im\ k = 0. \tag{23}$$

Multiplying (23) by a and subtracting from it, the equality (22) is multiplied by $\overline{b}$ using relation $|a(k)|^2 = 1 + |b(k)|^2$ for real k and from the definition of the function $r^+(k,t)$, we derive

$$\frac{\partial r^+(k,t)}{\partial t} = -\frac{i}{k}r^+(k,t) - \frac{1}{2ika^2(k)}\int_{-\infty}^{\infty} Gg^2 dx,\ Im\ k = 0. \tag{24}$$

For the classical KdV equation, this result was obtained in [5].

In the general case, the eigenvalues of the operator $L(t)y = -y'' + u(x,t)y$ depend on time, so differentiating the equalities

$$g(x,k_n,t) = B_n(t)f(x,k_n,t),\quad n = 1,2,...,N,$$

by t,

$$\frac{\partial g(x,k_n,t)}{\partial t} + \frac{\partial g}{\partial t}|_{k=k_n}\frac{dk_n}{dt} = \frac{dB_n}{dt}f(x,k_n,t)$$

$$+B_n\frac{\partial f(x,k_n,t)}{\partial t}+B_n\frac{\partial f}{\partial k}|_{k=k_n}\frac{dk_n}{dt}$$

those according to (15), we have

$$\frac{\partial g(x,i\chi_n,t)}{\partial t}=\frac{dB_n}{dt}f(x,i\chi_n,t)$$

$$+B_n\frac{\partial f(x,i\chi_n,t)}{\partial t}-i\frac{d\chi_n}{dt}\dot{a}(i\chi_n)h_n(x,t). \tag{25}$$

Similarly to the continuous spectrum, in the case of a discrete spectrum, we will look for the compatibility condition of the Lax pair in the form:

$$-g_{xx}(x,i\chi_n,t)+u(x,t)g(x,i\chi_n,t)=-\chi_n^2g(x,i\chi_n,t) \tag{26}$$

$$\frac{\partial g(x,i\chi_n,t)}{\partial t}=\frac{B_n}{2\chi_n^2}\left(v^2\frac{\partial f(x,i\chi_n,t)}{\partial x}-vv_xf(x,i\chi_n,t)\right)$$

$$+\left(-\frac{1}{2\chi_n}\int_{-\infty}^{x}Gg(x,i\chi_n,t)h_n(x,t)dx-\frac{B_n}{2\chi_n}\right)f(x,i\chi_n,t)$$

$$+\frac{1}{2\chi_nB_n}\int_{-\infty}^{x}Gg^2(x,i\chi_n,t)dxh_n(x,t). \tag{27}$$

Passing in this equality to the limit as $x\to\infty$ and using the asymptotics for $f(x,i\chi_n,t),\ h_n(x,t),\ u(x,t)$ we get:

$$\frac{dB_n}{dt}e^{-\chi_nt}-i\frac{d\chi_n}{dt}\dot{a}(i\chi_n)e^{\chi_nx}$$

$$=\frac{1}{2\chi_n^2}B_n(-\chi_n)e^{-\chi_nx}-\frac{B_ne^{-\chi_nx}}{2\chi_nB_n}\int_{-\infty}^{\infty}Gg(x,i\chi_n,t)h_n(x,t)dx$$

$$-\frac{1}{2\chi_n}B_ne^{-\chi_nx}+\frac{e^{\chi_nx}}{2\chi_nB_n}\int_{-\infty}^{\infty}Gg^2(x,i\chi_n,t)dx.$$

Indeed,

$$\frac{dB_n}{dt}=-\frac{B_n}{\chi_n}-\frac{1}{2\chi_n}\int_{-\infty}^{\infty}Gg(x,i\chi_n,t)h_n(x,t)dx, \tag{28}$$

$$i\frac{d\chi_n}{dt}\dot{a}(i\chi_n)=\frac{1}{2\chi_nB_n}\int_{-\infty}^{\infty}Gg^2(x,i\chi_n,t)dx,\quad n=1,2,...,N. \tag{29}$$

Let now $\Phi_n=\Phi_n(x,t)$ be the normalized eigenfunction of the operator $L(t)$ corresponding to the eigenvalue $\lambda_n=-\chi_n^2,\ n=1,2,...,N$, then according to the identity

$$g(x,i\chi_n,t)=b_n\Phi_n(x,t),$$

equality (29) can be rewritten as

$$i\dot{a}(i\chi_n)\frac{d\chi_n}{dt} = \frac{b_n^2}{2\chi_n B_n}\int_{-\infty}^{\infty} G\Phi_n^2(x,t)dx. \tag{30}$$

As shown in the monograph [3]

$$\dot{a}(i\chi_n) = -i\frac{b_n^2}{B_n},$$

therefore, equality (29) takes the form:

$$\frac{d\chi_n}{dt} = -\frac{1}{2\chi_n}\int_{-\infty}^{\infty} G\Phi_n^2(x,t)dx. \tag{31}$$

Thus, equality (24), (28), and (31) can be combined into the following main theorem.

THEOREM 4.1. *If the potential of the operator $L(t) = -\frac{d^2}{dx^2} + u(x,t)$ is a solution of equation (19) in the class of functions satisfying the conditions (4) then the scattering data of the operator $L(t)$ changes for t as follows:*

$$\frac{\partial r^+(k,t)}{\partial t} = -\frac{i}{k}r^+(k,t)$$
$$-\frac{1}{2ika^2(k)}\int_{\infty}^{\infty} G(x,t)g^2(x,i\chi_n,t)dx, \quad Im\ k=0,$$

$$\frac{dB_n(t)}{dt} = -\frac{B_n(t)}{\chi_n(t)} - \frac{1}{2\chi_n(t)}\int_{-\infty}^{\infty} G(x,t)g(x,i\chi_n,t)h_n(x,t)dx,$$

$$\frac{d\chi_n(t)}{dt} = -\frac{1}{2\chi_n(t)}\int G(x,t)\Phi_n^2(x,t)dx, \quad n=1,2,...,N,$$

where $\Phi_n(x,t)$ is the normalized eigenfunction of the operator $L(t)$ corresponding to the eigenvalue $\lambda_n = -\chi_n^2(t)$.

We will apply the result of Theorem 4.1 for

$$G(x,t) = 2\sum_{m=1}^{N}\frac{\partial}{\partial x}(\varphi_m\psi_m). \tag{32}$$

To apply the results of Theorem 4.1, we need to calculate the following integrals: For $m \neq n$

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\varphi_m\psi_m)\Phi_n^2 dx = 0. \tag{33}$$

For $m = n$, noticing that $\Phi_n = d_n\varphi_n$, taking into account (3), we get:

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(\varphi_n\psi_n)\Phi_n^2 dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (\varphi_n, \Phi_n W\{\Phi_n, \varphi_n\} + \psi_n\Phi_n W\{\Phi_n, \varphi_n\})\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \Phi_n^2 W\{\varphi_n, \psi_n\}\, dx = \frac{\omega_n(t)}{2}. \tag{34}$$

Combining (33) and (34) by Theorem 4.1, we infer that

$$\frac{d\chi_n(t)}{dt} = -\frac{\omega_n(t)}{2\chi_n(t)}, \quad n = 1, 2, ..., N. \tag{35}$$

By the definition of the function ψ_m and the asymptotics of the Jost solutions,

$$\psi_m \to -\frac{\omega_m}{2\chi_m c_m^-}\exp(-\chi_m x), \quad \text{in } x \to -\infty,$$
$$\psi_m \to \frac{\omega_m}{2\chi_m c_m^+}\exp(\chi_m x), \quad \text{in } x \to +\infty, \tag{36}$$

where c_m^+, c_m^- are determined from the equalities

$$\varphi_m(x,t) = c_m^+ f(x, i\chi_m, t),$$
$$\varphi_m(x,t) = c_m^- g(x, i\chi_m, t), \quad m = 1, 2, ..., N. \tag{37}$$

Since the function h_n is a solution to the equation $L(t)h_n(x,t) = \lambda_n h_n(x,t)$,

$$h_n(x,t) = \alpha_n(t)\psi_n(x,t) + \beta_n(t)g(x, i\chi_n, t). \tag{38}$$

According to (3) and (18)

$$\alpha_n(t) = \frac{2\chi_n c_n^-(t)B_n(t)}{\omega_n(t)}, \quad n = 1, 2, ..., N, \tag{39}$$

besides

$$W\{h_n, \psi_n\} = \frac{\beta_n\omega_n}{c_n^-}, \quad n = 1, 2, ..., N. \tag{40}$$

Now we will calculate the next integral for $m \neq n$:

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(\varphi_m\psi_m)g_n h_n dx$$

$$= -\frac{1}{2(\lambda_n - \lambda_m)}\left(W\{g_n, \psi_n\} W\{h_n, \psi_m\}\right)|_{-\infty}^{\infty},$$

here we use asymptotic formulas (16), (17), and (36) and obtain:

$$\left(W\{g_n, \varphi_m\} W\{h_n, \varphi_m\}\right)|_{-\infty}^{\infty} = 0.$$

For $m = n$ using (40), (30) we have

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(\varphi_n \psi_n) g_n h_n dx = \frac{i\dot{a}(i\chi_n)\beta_n \omega_n B_n}{2}.$$

Thus,

$$\frac{dB_n(t)}{dt} = \left(-\frac{1}{2\chi_n(t)} - \frac{1}{2\chi_n(t)} i\dot{a}(i\chi_n)\beta_n(t)\omega_n(t)\right) B_n(t), \qquad (41)$$

$n = 1, 2, ..., N.$

Similarly, using the definition of Jost solutions, and the asymptotic formulas of $\varphi_m(x,t)$, $\psi_m(x,t)$ we have

$$\int_{-\infty}^{\infty} 2\sum_{m=1}^{N} \frac{\partial}{\partial x}\left(\varphi_m \psi_m\right) g^2 dx = -\frac{2\omega_m k^2}{\chi_m(t)(k^2 - \chi_m^2(t))} a(k)\bar{b}(k),$$

therefore,

$$\frac{\partial r^+(k,t)}{\partial t} = \left(-\frac{i}{k} + \sum_{m=1}^{N} \frac{k\omega_n}{i\chi_m(t)\left(k^2 + \chi_m^2(t)\right)}\right) r^+(k,t), \;\; Im\, k = 0. \; (42)$$

The results obtained in (35), (41), and (42) can be combined to the following theorem.

Theorem 4.2. *If the functions $u(x,t), v(x,t), \varphi_1, ..., \varphi_m, \psi_1, ..., \psi_m$ are solutions to problem (1) - (5) then the scattering data of the operator $L(t) \equiv -\frac{d^2}{dx^2} + u(x,t)$ change with respect to t as follows:*

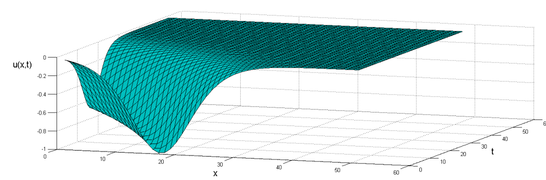
$$\frac{\partial r^+(k,t)}{\partial t} = \left(-\frac{i}{k} + \sum_{m=1}^{N} \frac{k\omega_n}{i\chi_m(t)\left(k^2 + \chi_m^2(t)\right)}\right) r^+(k,t), \quad Im\ k = 0,$$

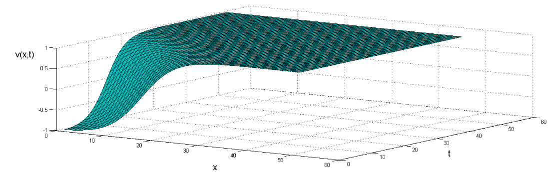
$$\frac{dB_n(t)}{dt} = \left(-\frac{1}{2\chi_n(t)} - \frac{1}{2\chi_n(t)} i\dot{a}(i\chi_n)\beta_n(t)\omega_n(t)\right) B_n(t),$$

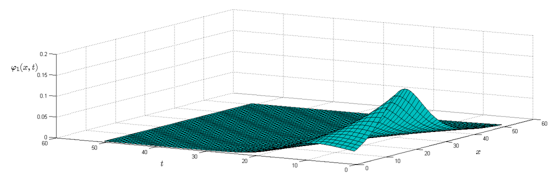
$$\frac{d\chi_n(t)}{dt} = -\frac{\omega_n}{2\chi_n(t)}, \quad n = 1, 2, ..., N.$$

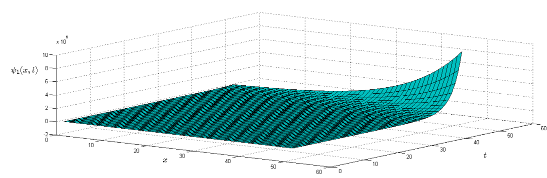
Note that, for the classical KdV equation this result was obtained in the work [5]. The resulting equality completely determines the evolution of the scattering data, which makes it possible to apply the IST method to solve problem (1)-(5).

EXAMPLE 4.1. Let
$$u(x,0) = -\frac{2}{ch^2 x}.$$
In this case,
$$N = 1, \quad r^+(k,0) = \frac{b(-k,0)}{a(k,0)} = 0, \quad B_1(0) = 1, \quad \chi_1(0) = 1.$$
Applying Theorem 4.1, we get:
$$r^+(k,t) = 0, \ B_1(t) = e^{2\gamma_1(t)}, \ \chi_1^2(t) = 1 - \int_0^t \omega_1(\tau) d\tau,$$
where
$$\gamma_1(t) = \frac{1}{2} \int_0^t \left(-\frac{1}{\chi_1} - \frac{\beta_1 \omega_1}{4\chi_1^2} \right) d\tau.$$
Solving the inverse problem, we get
$$u(x,t) = -\frac{2\chi_1^2}{ch^2(\chi_1 x - \gamma_1)}.$$
Therefore, we find
$$f(x, i\chi_1, t) = \frac{e^{-\gamma_1}}{2ch(\chi_1 x - \gamma_1)}, \quad g(x, i\chi_1, t) = \frac{e^{\gamma_1}}{2ch(\chi_1 x - \gamma_1)},$$
$$h_1(x,t) = 2e^{\gamma_1} \left(sh(\chi_1 x - \gamma_1) + \frac{x\chi_1}{ch(\chi_1 x - \gamma_1)} \right).$$
Assuming
$$\varphi_1(x,t) = \frac{e^{\gamma_1}}{2ch(\chi_1 x - \gamma_1)},$$
from equality (37), (38), (39) we obtain,
$$\psi_1(x,t) = \frac{\omega_1 e^{-\gamma_1}}{\chi_1} \left(sh(\chi_1 x - \gamma_1 + \frac{\chi_1 x}{ch(\chi_1 x - \gamma_1} - \frac{\beta_1}{4ch(\chi_1 x - \gamma_1)} \right).$$
After some calculations, as a result we find
$$v = th(\chi_1 x - \gamma_1).$$
Graphics of the function $u(x,t)$, $v(x,t)$, $\varphi_1(x,t)$ and $\psi_1(x,t)$
where
$$\omega_1(t) = \frac{1}{(t+1)^2}, \ \beta_1(t) = 4(t+1)^{\frac{3}{2}}, \ \chi_1(t) = \frac{1}{\sqrt{t+1}},$$
$$\gamma_1(t) = \frac{2}{3} - \frac{2}{3}(t+1)^{\frac{3}{2}}$$
are given below.









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Conclusions

Without additional conditions (4), as in the work of [5], the eigenvalues of the operator $L(t)$ in the considered model depend on t (moving eigenvalues). In this case, some moving eigenvalues may enter the continuous spectrum, or two distinct eigenvalues may merge. As a result, the number of eigenvalues of the operator $L(t)$ decreases (some disappear). Over time, the merged eigenvalues may split again, and the eigenvalues that entered the continuous spectrum may reappear as discrete eigenvalues. This phenomenon gives rise to the effect known as the creation and annihilation of solitons [6].

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