International Journal of Applied Mathematics

Volume 38 No. 3 2025, 351–367
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v38i3.3

UNIQUENESS AND CONVERGENCE ANALYSIS OF THE FRACTIONAL VOLTERRA-FREDHOLM MODEL

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Received: March 7, 2025

Abstract

The multi-term fractional-order Volterra-Fredholm models (FV-FMs) were the main topic of this paper. The Banach contraction principle is applied to establish the uniqueness of the solution for multi-term FV-FMs under certain conditions. Additionally, the solution's convergence is examined and demonstrated. To illustrate the theorems' application, a few examples are proposed.

Math. Subject Classification: 34A09, 34A08, 34D20

Key Words and Phrases: Volterra-Fredholm model, convergence analysis, Caputo fractional derivative, fixed point

1. Introduction

Fractional calculus has drawn a great deal of interest from academics in the last several decades from a variety of mathematical and scientific fields. Many researchers have recently examined the existence and uniqueness of solutions to these equations. For example, [4] demonstrated the existence results for fractional integro-differential equations with non-local conditions using resolvent operators. Additionally, [4, 7, 10, 14, 17, 20] and several others have recently investigated the presence of solutions to other kinds of fractional differential equations as well as fractional integro-differential equations of boundary value problems and initial value issues. Moreover, numerous engineering, social science, and management disciplines have documented the use of these equations.

In other words, fractional calculus is frequently used to explain a wide range of significant scientific phenomena, including chemical reactions, fluid mechanics, and chaotic systems [12], [21], and [22]. In the Social Sciences [2, 6, 11, 16], Economics [18], Finance [8], and so on. Fractional operators are a more favorable way to describe many physical phenomena since they take into consideration how these phenomena evolve. For some of these equations, it can be challenging to discover analytical solutions, though. Consequently, a numerical method is required. Numerical methods for approximating the solutions to this class of problems have been studied in the last few decades, see [1, 3, 9, 13, 15]. The Volterra type equations constitute a unique class of these equations, and their applications include heat transfer problems, nanohydrodynamics, mass diffusion processes, neutron diffusion, coexistence of biological species with decreasing and increasing rates of growth, and electromagnetic theory [19].

The multi-term Volterra-Fredholm fractional equation of the following form was examined in this paper:

$$\begin{aligned} D^{\beta}\Phi(\omega) &= \sum_{j=0}^{k} c_j(\omega) D^{\delta_j}\Phi(\omega) + h(\omega) + \int_0^{\rho} \chi(\rho,\tau) H(\Phi(\tau)) d\tau \\ &\quad + \int_0^1 A(\rho,\tau) B(\Phi(\tau)) d\tau, \qquad (1) \\ \Phi^{(n)}(0) &= d_n,\ n = 0,1,2,\ldots,m-1, \qquad (2) \end{aligned}$$

where D^{β} is the Caputo differential fractional operator, $m-1 < \delta_0 < \delta_1 < \cdots < \delta_j < \beta \leq m, m, k \in \mathbb{N}$, $\Phi : Q \longrightarrow \mathbb{R}, Q = [0,1]$ is a continuous function which needs to be determined, $c_j, h : Q \longrightarrow \mathbb{R}$ be given continuous functions, $\chi, A : Q \times Q \longrightarrow \mathbb{R}$ be the continuous kernel of integration, $H, B : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz functions.

2. **Preliminaries**

Definition 2.1. [19] Let G be a metric space, a mapping $\Omega : G \longrightarrow G$ is a contraction if $\exists\, L \in [0,1)$ such that $\|\omega - \Omega u\| \leq L\|\omega - u\|$, $\forall\, \omega, u \in G$.

Definition 2.2. [19] Given a map $\Omega : A \longrightarrow B$, every solution u of the equation

$$\Omega u = u,$$

is called a fixed point of Ω.

Definition 2.3. [19] Let X be a complete metric space, then each contraction mapping $\Omega : G \longrightarrow G$ bas a unique fixed point u of Ω in G, that is, $\Omega u = u$.

Definition 2.4. [19] The Riemann-Liouville fractional integral of order β of a function Φ is defined as

$$I^{\beta}\Phi(\omega) = \frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega - v)^{\beta - 1} \Phi(v) dv, \quad \omega > 0,\ \beta \in \mathbb{R}^+, \qquad (3)$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.5. [19] The Riemann-Liouville fractional derivative of order $\beta > 0$ of a function Φ is defined as

$$D^{\beta}\Phi(\omega) = D^{q}I^{q-\beta}\Phi(\omega) \quad (q-1 < \beta \leq q,\ q \in \mathbb{N})$$
$$= \frac{d^q}{dt^q}\left(\frac{1}{\Gamma(q-\beta)}\int_0^{\omega}(\omega - v)^{q-\beta-1}\Phi(v)dv\right).$$

Definition 2.6. [19] The fractional derivative of $\Phi(\omega)$ in the Caputo sense is defined by

$$D^{\beta}\Phi(\omega) = I^{q-\beta}D^{q}\Phi(\omega)$$
$$= \frac{1}{\Gamma(q-\beta)}\int_0^{\omega}(\omega - v)^{q-\beta-1}\frac{d^q\Phi(v)}{dv^q}dv, \quad q-1 < \beta \leq q.$$

The following properties hold:

i. $I^{\beta}D^{\beta}\Phi(\omega) = \Phi(\omega) - \sum_{n=0}^{q-1}\frac{\Phi^{(n)}(0)}{n!}\omega^n, q-1 < \beta \leq q,$

ii. $I^{\beta}D^{\gamma}\Phi(\omega) = I^{\beta-\gamma}\Phi(\omega), 0 < \gamma < \beta$, and $q-1 < \beta \leq q, q \in \mathbb{N}$,

iii. $I^{\alpha}\Phi(\omega) = \frac{\omega^{\alpha}}{\Gamma(\alpha+1)}$, where $\Phi(\omega) = 1, \omega \in [0,1]$.

3. Main results

In this work, we denote by

(i) $\|\cdot\|_{\infty}$ the sup norm on $C(Q,\mathbb{R})$, i.e for $c \in C(Q,\mathbb{R}), \|c\|_{\infty} = \sup_{\omega\in Q}|c(\omega)|$.

(ii) $\|\cdot\|_{\infty}$ the sup norm on $C(Q,\mathbb{R})$, i.e for $h \in C(Q,\mathbb{R}), \|h\|_{\infty} = \sup_{\omega\in Q}|h(\omega)|$.

We make the following hypotheses:

(p_1) there exist constants $\Psi, \Psi_B > 0$ such that for any $\Phi_1, \Phi_2 \in C(Q,\mathbb{R})$ we have

$$|H(\Phi_1(\omega)) - H(\Phi_2(\omega))| \leq \Psi\|\Phi_1 - \Phi_2\|_{\infty}\, \omega \in Q,$$
$$|B(\Phi_1(\omega)) - B(\Phi_2(\omega))| \leq \Psi_B\|\Phi_1 - \Phi_2\|_{\infty}\, \omega \in Q,$$

(p_2) there exists a constant Π, Π_A such that

$$\Pi = \sup_{\rho\in[0,1]}\int_0^{\rho}|\chi(\rho,\tau)|d\tau < \infty,\ \Pi_A = \sup_{\rho\in[0,1]}\int_0^{1}|A(\rho,\tau)|d\tau < \infty.$$

Lemma 3.1. *Let $\Phi : Q \longrightarrow \mathbb{R}$ and $g : Q \longrightarrow \mathbb{R}$ be continuous functions. Then, a function Φ is a solution to the model* (1)-(2) *if, and only if,*

$$\begin{aligned}
&\Phi(\omega) \\
=\ & \sum_{n=0}^{m-1} \frac{d_n}{n!}\omega^n + \sum_{j=0}^{k} c_j(\omega) I^{\beta-\delta_j}\Phi(s) + I^\beta h(\omega) \qquad (4)\\
&+ I^\beta \left(\int_0^\rho \chi(\rho,\tau) H(\Phi(\tau)) d\tau \right) + I^\beta \left(\int_0^1 A(\rho,\tau) B(\Phi(\tau)) d\tau \right).
\end{aligned}$$

Proof. Applying equation (3) on equation (1) and using properties (i), (ii) together with the condition (2), we have

$$\begin{aligned}
& I^\beta \left(D^\beta \Phi(\omega) \right) \\
=& I^\beta \left(\sum_{j=0}^{k} c_j(\omega) D^{\delta_j}\Phi(\omega) \right) + I^\beta (h(\omega)) \\
& + I^\beta \left(\int_0^\rho \chi(\rho,\tau) H(\Phi(\tau)) d\tau \right) + I^\beta \left(\int_0^\rho \chi(\rho,\tau) H(\Phi(\tau)) d\tau \right) \\
=& \sum_{j=0}^{k} c_j(\omega) I^\beta \left(D^{\delta_j}\Phi(\omega) \right) + I^\beta (h(\omega)) \\
& + I^\beta \left(\int_0^\rho \chi(\rho,\tau) H(\Phi(\tau)) d\tau \right) + I^\beta \left(\int_0^1 A(\rho,\tau) B(\Phi(\tau)) d\tau \right) \\
=& \sum_{n=0}^{m-1} \frac{\Phi^{(n)}(0)}{n!}\omega^n + \sum_{j=0}^{k} c_j(\omega) I^{\beta-\delta_j}\Phi(\omega) + I^\beta g(\omega) \\
& + I^\beta \left(\int_0^\rho \chi(\rho,\tau) H(\Phi(\tau)) d\tau \right) + I^\beta \left(\int_0^1 A(\rho,\tau) B(\Phi(\tau)) d\tau \right) \\
=& \sum_{n=0}^{m-1} \frac{d_n}{n!}\omega^n + \sum_{j=0}^{k} c_j(\omega) I^{\beta-\delta_j}\Phi(\omega) + I^\beta h(\omega) \\
& + I^\beta \left(\int_0^\rho \chi(\rho,\tau) H(\Phi(\tau)) d\tau \right) + I^\beta \left(\int_0^1 A(\rho,\tau) B(\Phi(\tau)) d\tau \right).
\end{aligned}$$

Thus, Φ solves model (1)-(2) if, and only if, Φ solves (4).

Theorem 3.2. *Assume that* (p_1) *and* (p_2) *bold, and if*

$$\left(\sum_{j=0}^{k}\frac{\|c_j\|_\infty}{\Gamma\left(\beta-\delta_j+1\right)}+\frac{(\Pi+\Pi_A)\Psi}{\Gamma(\beta+1)}\right)<1, \tag{5}$$

then there is a unique solution $\Phi\in C(Q,\mathbb{R})$ *to the model* (1)-(2).

Proof. Let T be an operator such that $T:C(Q,\mathbb{R})\longrightarrow C(Q,\mathbb{R})$ defined from equation (4) as

$$\begin{aligned}(T\Phi)(\omega)=&\sum_{n=0}^{m-1}\frac{d_n}{n!}\omega^n+\sum_{j=0}^{k}\frac{1}{\Gamma\left(\beta-\delta_j\right)}\int_0^\omega(\omega-\rho)^{\beta-\delta_j-1}c_j(\rho)\Phi(\rho)d\rho\\&+\frac{1}{\Gamma(\beta)}\int_0^\omega(\omega-\rho)^{\beta-1}g(\rho)d\rho\\&+\frac{1}{\Gamma(\beta)}\int_0^\omega(\omega-\rho)^{\beta-1}\left(\int_0^\rho\chi(\rho,\tau)H(\Phi(\tau))d\tau\right)d\rho\\&+\frac{1}{\Gamma(\beta)}\int_0^\omega(\omega-\rho)^{\beta-1}\left(\int_0^1 A(\rho,\tau)B(\Phi(\tau))d\tau\right)d\rho.\end{aligned} \tag{6}$$

The objective here is to apply the Banach contraction principle. To do that, we will show that T is a contraction.

First, we note that T is well defined. Indeed, since

$\omega\longmapsto\sum_{n=0}^{m-1}\frac{d_n}{n!}\omega^n,\ \ \omega\longmapsto\sum_{j=0}^{k}c_j(\omega)\left(I^{\beta-\delta_j}\Phi\right)(\omega)$

$\omega\longmapsto\left(I^{\beta}h\right)(\omega),\ \omega\longmapsto\int_0^\rho\chi(\rho,\tau)H(\Phi(\tau))d\tau,$

$\omega\longmapsto\int_0^1 A(\rho,\tau)B(\Phi(\tau))d\tau$

are continuous, the right hand side of equation (7) is well defined and $\omega\longmapsto(T\Phi)(\omega)$ is continuous. Thus, for $\Phi\in C(Q,\mathbb{R}),T\Phi$ is also in $C(Q\mathbb{R})$.

Let $\Phi_1, \Phi_2 \in C(Q, \mathbb{R})$ and let $\omega \in [0,1]$. By the definition of T we have

$$
\begin{aligned}
&\left|\left(T\Phi_1\right)(\omega) - \left(T\Phi_2\right)(\omega)\right| \\
=&\left| \sum_{j=0}^{k} \frac{1}{\Gamma\left(\beta-\delta_j\right)} \int_0^{\omega} (\omega-\rho)^{\beta-\delta_j-1} c_j(\rho) \left(\Phi_1(\rho) - \Phi_2(\rho)\right) d\rho \right. \\
&+\frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} \left(\int_0^{\rho} \chi(\rho,\tau) \left(H\left(\Phi_1(\tau)\right) - H\left(\Phi_2(\tau)\right)\right) d\tau \right) d\rho \Bigg| \\
&\left. +\frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} \left(\int_0^{1} A(\rho,\tau) \left(B\left(\Phi_1(\tau)\right) - B\left(\Phi_2(\tau)\right)\right) d\tau \right) d\rho \right| \\
\leq& \left| \sum_{j=0}^{k} \frac{1}{\Gamma\left(\beta-\delta_j\right)} \int_0^{\omega} (\omega-\rho)^{\beta-\delta_j-1} c_j(\rho) \left(\Phi_1(\rho) - \Phi_2(\rho)\right) d\rho \right| \\
&+ \left| \frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} \left(\int_0^{\rho} \chi(\rho,\tau) \left(H\left(\Phi_1(\tau)\right) - H\left(\Phi_2(\tau)\right)\right) d\tau \right) d\rho \right| \\
&+ \left| \frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} \left(\int_0^{1} A(\rho,\tau) \left(B\left(\Phi_1(\tau)\right) - B\left(\Phi_2(\tau)\right)\right) d\tau \right) d\rho \right| \\
\leq& \sum_{j=0}^{k} \frac{1}{\Gamma\left(\beta-\delta_j\right)} \int_0^{\omega} (\omega-\rho)^{\beta-\delta_j-1} \left|c_j(\rho)\right| \left|\Phi_1(\rho) - \Phi_2(\rho)\right| d\rho \\
&+ \frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} \left(\int_0^{\rho} \left|\chi(\rho,\tau)\right| \left|H\left(\Phi_1(\tau)\right) - H\left(\Phi_2(\tau)\right)\right| d\tau \right) d\rho \\
&+ \frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} \left(\int_0^{1} \left|A(\rho,\tau)\right| \left|B\left(\Phi_1(\tau)\right) - B\left(\Phi_2(\tau)\right)\right| d\tau \right) d\rho \\
\leq& \sum_{j=0}^{k} \frac{\left\|c_j\right\|_{\infty} \left\|\Phi_1 - \Phi_2\right\|_{\infty}}{\Gamma\left(\beta-\delta_j\right)} \int_0^{\omega} (\omega-\rho)^{\beta-\delta_j-1} d\rho \\
&+ \frac{\Pi\Psi \left\|\Phi_1 - \Phi_2\right\|_{\infty}}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} d\rho \\
&+ \frac{\Pi_A\Psi \left\|\Phi_1 - \Phi_2\right\|_{\infty}}{\Gamma(\beta)} \int_0^{\omega} (\omega-\rho)^{\beta-1} d\rho \\
\leq& \sum_{j=0}^{k} \frac{\left\|c_j\right\|_{\infty} \left\|\Phi_1 - \Phi_2\right\|_{\infty}}{\Gamma\left(\beta-\delta_j+1\right)} \omega^{\beta-\delta_j} + \frac{\Pi\Psi \left\|\Phi_1 - \Phi_2\right\|_{\infty}}{\Gamma(\beta+1)} \omega^{\beta} + \frac{\Pi_A\Psi \left\|\Phi_1 - \Phi_2\right\|_{\infty}}{\Gamma(\beta+1)} \omega^{\beta}.
\end{aligned}
$$

Thus,

$$\|T\Phi_1 - T\Phi_2\|_\infty \leq \left(\sum_{j=0}^{k} \frac{\|c_j\|_\infty}{\Gamma(\beta - \delta_j + 1)} + \frac{(\Pi + \Pi_A)\Psi}{\Gamma(\beta+1)}\right) \|\Phi_1 - \Phi_2\|_\infty .$$

We conclude that T is a contraction, since by equation (5),

$$\left(\sum_{j=0}^{k} \frac{\|c_j\|_\infty}{\Gamma(\beta - \delta_j + 1)} + \frac{(\Pi + \Pi_A)\Psi}{\Gamma(\beta+1)}\right) < 1.$$

By the Banach contraction principle, T has a unique solution Φ in $C(Q, \mathbb{R})$.

Theorem 3.3. *If the solution is convergent, then it converges to the exact solution of the Model* (1)-(2).

Proof. Let ϑ_u, ϑ_v be arbitrary partial sums with $v \leq u$. We show that ϑ_u is a Cauchy sequence. Let $\vartheta_u = \sum_{j=0}^{u} \Phi_j(\omega)$ and $\vartheta_v = \sum_{j=0}^{v} \Phi_j(\omega)$. Since $v \leq u$, then, we have from equation (4):

$$\begin{aligned}
&\vartheta_u - \vartheta_v = \sum_{j=v+1}^{u} \Phi_j(\omega) \\
&= \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \int_0^{\omega} (\omega - \rho)^{\beta - \delta_j - 1} c_j(\rho) \sum_{j=v+1}^{u} \Phi_j(\rho) d\rho \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega - \rho)^{\beta - 1} \left(\int_0^{\rho} \chi(\rho, \tau) H \left(\sum_{j=v+1}^{u} \Phi_j(\tau) \right) d\tau \right) d\rho \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{\omega} (\omega - \rho)^{\beta - 1} \left(\int_0^{1} A(\rho, \tau) B \left(\sum_{j=v+1}^{u} \Phi_j(\tau) \right) d\tau \right) d\rho.
\end{aligned} \tag{7}$$

Let

$$H\left(\sum_{j=v+1}^{u} \Phi_j(\tau)\right) = \sum_{j=v+1}^{u} G_j(\tau)$$

and

$$B\left(\sum_{j=v+1}^{u} \Phi_j(\tau)\right) = \sum_{j=v+1}^{u} U_j(\tau),$$

then equation (7) becomes

$$
\begin{aligned}
|\vartheta_u-\vartheta_v| &= \left|\sum_{j=v+1}^{u}\Phi_j(\omega)\right| \\
&= \Bigg|\sum_{j=0}^{k}\frac{1}{\Gamma\left(\beta-\delta_j\right)}\int_0^{\omega}(\omega-\rho)^{\beta-\delta_j-1}c_j(\rho)\sum_{j=v+1}^{u}\Phi_j(\rho)d\rho \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^{\omega}(\omega-\rho)^{\beta-1}\left(\int_0^{\rho}\chi(\rho,\tau)\sum_{j=v+1}^{u}G_j(\tau)d\tau\right)d\rho\Bigg| \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^{\omega}(\omega-\rho)^{\beta-1}\left(\int_0^{1}A(\rho,\tau)\sum_{j=v+1}^{u}U_j(\tau)d\tau\right)d\rho\Bigg| .
\end{aligned}
\tag{8}
$$

If we let

$$
\sum_{j=v}^{u-1}\Phi_j(\omega)=\vartheta_{u-1}-\vartheta_{v-1},\sum_{j=v}^{u-1}G_j(t)=H\left(\vartheta_{u-1}\right)-H\left(\vartheta_{v-1}\right),
$$

$$
\sum_{j=v}^{u-1}U_j(t)=B\left(\vartheta_{u-1}\right)-B\left(\vartheta_{v-1}\right)
$$

in equation (8), then we have

$$
\begin{aligned}
&\|\vartheta_u-\vartheta_v\|_{\infty} \\
&\le \max_{\forall\omega\in Q}\Bigg(\Bigg|\sum_{j=0}^{k}\frac{1}{\Gamma\left(\beta-\delta_j\right)}\int_0^{\omega}(\omega-\rho)^{\beta-\delta_j-1}c_j(\rho)\left(\vartheta_{u-1}-\vartheta_{v-1}\right)d\rho \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^{\omega}(\omega-\rho)^{\beta-1}\left(\int_0^{\rho}\chi(\rho,\tau)\left(H\left(\vartheta_{u-1}\right)-H\left(\vartheta_{v-1}\right)\right)d\tau\right)d\rho\Bigg|\Bigg) \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^{\omega}(\omega-\rho)^{\beta-1}\left(\int_0^{1}A(\rho,\tau)\left(B\left(\vartheta_{u-1}\right)-B\left(\vartheta_{v-1}\right)\right)d\tau\right)d\rho\Bigg|\Bigg) \\
&\le \max_{\forall\omega\in Q}\Bigg(\sum_{j=0}^{k}\frac{1}{\Gamma\left(\beta-\delta_j\right)}\int_0^{\omega}(\omega-\rho)^{\beta-\delta_j-1}\left|c_j(\rho)\right|\left|\vartheta_{u-1}-\vartheta_{v-1}\right|d\rho \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^{\omega}(\omega-\rho)^{\beta-1}\left(\int_0^{\rho}\left|\chi(\rho,\tau)\right|\left|H\left(\vartheta_{u-1}\right)-H\left(\vartheta_{v-1}\right)\right|d\tau\right)d\rho\Bigg) \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^{\omega}(\omega-\rho)^{\beta-1}\left(\int_0^{1}\left|A(\rho,\tau)\right|\left|B\left(\vartheta_{u-1}\right)-B\left(\vartheta_{v-1}\right)\right|d\tau\right)d\rho\Bigg)
\end{aligned}
$$

$$\leq \sum_{j=0}^{k} \frac{\|c_j\|_\infty \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty}{\Gamma(\beta - \delta_j)} \int_0^\omega (\omega - \rho)^{\beta - \delta_j - 1} d\rho$$
$$+ \frac{\Pi\Psi \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty}{\Gamma(\beta)} \int_0^\omega (\omega - \rho)^{\beta - 1} d\rho$$
$$+ \frac{\Pi_A\Psi \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty}{\Gamma(\beta)} \int_0^\omega (\omega - \rho)^{\beta - 1} d\rho$$
$$\leq \left(\sum_{j=0}^{k} \frac{\|c_j\|_\infty}{\Gamma(\beta - \delta_j + 1)} + \frac{(\Pi + \Pi_A)\Psi}{\Gamma(\beta + 1)} \right) \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty .$$

Thus,

$$\|\vartheta_u - \vartheta_v\|_\infty \leq \Upsilon \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty , \tag{9}$$

where $\Upsilon = \sum_{j=0}^{k} \frac{\|c_c\|_\infty}{\Gamma(\beta - \delta_j + 1)} + \frac{(\Pi + \Pi_A)\Psi}{\Gamma(\beta + 1)}$. Observe that, from equation (9):

$$\begin{aligned} &\|\vartheta_u - \vartheta_v\|_\infty \\ &\leq \Upsilon \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty \leq \Upsilon \|\vartheta_{u-2} - \vartheta_{v-2}\|_\infty \leq \ldots \leq \Upsilon \|\vartheta_1 - \vartheta_0\|_\infty . \end{aligned}$$

Also from equation (9),

$$\begin{aligned} \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty &\leq \Upsilon \|\vartheta_{u-2} - \vartheta_{v-2}\|_\infty , \\ \|\vartheta_{u-2} - \vartheta_{v-2}\|_\infty &\leq \Upsilon \|\vartheta_{u-3} - \vartheta_{v-3}\|_\infty , \ \ldots . \end{aligned}$$

Therefore,

$$\begin{aligned} &\|\vartheta_u - \vartheta_v\|_\infty \\ &\leq \Upsilon \|\vartheta_{u-1} - \vartheta_{v-1}\|_\infty \leq \Upsilon^2 \|\vartheta_{u-2} - \vartheta_{v-2}\|_\infty \leq \ldots \leq \Upsilon^v \|\vartheta_1 - \vartheta_0\|_\infty . \end{aligned} \tag{10}$$

Let $u = v + 1$, accordingly in equation (10), then we have

$$\begin{aligned} &\|\vartheta_u - \vartheta_v\|_\infty \\ &\leq \Upsilon \|\vartheta_v - \vartheta_{v-1}\|_\infty \leq \Upsilon^2 \|\vartheta_{v-1} - \vartheta_{v-2}\|_\infty \leq \ldots \leq \Upsilon^v \|\vartheta_1 - \vartheta_0\|_\infty . \end{aligned}$$

That is,

$$\|\vartheta_p - \vartheta_q\|_\infty \leq \Upsilon^v \|\vartheta_1 - \vartheta_0\|_\infty , \tag{11}$$

$\|\vartheta_u - \vartheta_v\|_\infty$ can be written as follows

$$\begin{aligned}
&\|\vartheta_u - \vartheta_v\|_\infty \\
=&\|\vartheta_{v+1} - \vartheta_v + \vartheta_{v+2} - \vartheta_{v+1} + \vartheta_{v+3} - \vartheta_{v+2} + \vartheta_{v+4} - \vartheta_{v+3} \\
&+ \cdots + \vartheta_{v+(u-v-2)} - \vartheta_{v+(u-v-1)} + \vartheta_{v+(u-v)} - \vartheta_{v+(u-v-1)}\|_\infty \\
=&\|\vartheta_{v+1} - \vartheta_v + \vartheta_{v+2} - \vartheta_{v+1} + \vartheta_{v+3} - \vartheta_{v+2} + \vartheta_{v+4} - \vartheta_{v+3} \\
&+ \cdots + \vartheta_{u-2} - \vartheta_{u-1} + \vartheta_u - \vartheta_{u-1}\|_\infty \\
\leq& \|\vartheta_{v+1} - \vartheta_v\|_\infty + \|\vartheta_{v+2} - \vartheta_{v+1}\|_\infty + \|\vartheta_{v+3} - \vartheta_{v+2}\|_\infty + \\
&\|\vartheta_{v+4} - \vartheta_{v+3}\|_\infty + \cdots + \|\vartheta_{u-2} - \vartheta_{u-1}\|_\infty + \|\vartheta_u - \vartheta_{u-1}\|_\infty .
\end{aligned} \tag{12}$$

From equation (11), let $u = v + 1$, then

$$\begin{aligned}
\|\vartheta_{v+1} - \vartheta_v\|_\infty &\leq \Upsilon^v \|\vartheta_1 - \vartheta_0\|_\infty \\
\|\vartheta_{v+2} - \vartheta_{v+1}\|_\infty &\leq \Upsilon^{v+1} \|\vartheta_1 - \vartheta_0\|_\infty \\
\|\vartheta_{v+3} - \vartheta_{v+2}\|_\infty &\leq \Upsilon^{v+2} \|\vartheta_1 - \vartheta_0\|_\infty \\
&\vdots \\
\|\vartheta_u - \vartheta_{u-1}\|_\infty &\leq \Upsilon^{u-1} \|\vartheta_1 - \vartheta_0\|_\infty .
\end{aligned}$$

Therefore, equation (12) can be written as

$$\begin{aligned}
\|\vartheta_u - \vartheta_v\|_\infty &\leq \left(\Upsilon^v + \Upsilon^{v+1} + \Upsilon^{v+2} + \cdots + \Upsilon^{u-1}\right) \|\vartheta_1 - \vartheta_0\|_\infty \\
&= \Upsilon^v \left(1 + \Upsilon^1 + \Upsilon^2 + \cdots + \Upsilon^{u-v-1}\right) \|\vartheta_1 - \vartheta_0\|_\infty .
\end{aligned}$$

By the geometric series, let $q = u - v - 1$, this implies,

$$\|\vartheta_u - \vartheta_v\|_\infty \leq \Upsilon^v \left(\frac{1 - \Upsilon^{u-v}}{1 - \Upsilon}\right) \|\vartheta_1 - \vartheta_0\|_\infty ,$$

since $0 < \Upsilon < 1$, this means $1 - \Upsilon^{u-v} < 1$, then

$$\|\vartheta_u - \vartheta_v\|_\infty \leq \frac{\Upsilon^v}{1 - \Upsilon} \|\Phi_1\|_\infty .$$

But $|\Phi_1(\omega)| < \infty$ and $\lim_{v \longrightarrow \infty} \frac{\Upsilon^v}{1-\Upsilon} = 0$, since $\Upsilon^v \longrightarrow 0$ as $v \longrightarrow \infty$. Therefore, $\|\vartheta_u - \vartheta_v\|_\infty \longrightarrow 0$ as $v \longrightarrow \infty$. We conclude that ϑ_u is a Cauchy sequence in $C[0,1]$. Therefore, $\lim_{n \longrightarrow \infty} \Phi_n = \Phi$. Thus, the solution is convergent.

4. Applications

Example 1. Consider the Volterra-Fredholm model

$$\begin{aligned} D^{\frac{3}{4}}\Phi(\omega) &= \frac{6\omega^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} - \frac{\omega^2 e^{\omega}}{5}\Phi(\omega) + \int_0^{\omega} e^{\omega}\tau\Phi(\tau)d\tau + \int_0^1 \omega\tau\Phi(\tau)d\tau, \\ \Phi(0) &= 0, \end{aligned}$$

the exact solution is $\Phi(\omega) = \omega^3$.

From Theorem 3.2, we have

$$\begin{aligned} &\left|\left(T\Phi_2\right)(\omega) - \left(T\Phi_1\right)(\omega)\right| \\ =&\left| -\frac{1}{5\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \rho^2 e^{\rho}\left(\Phi_2(\rho)-\Phi_1(\rho)\right) d\rho \right. \\ &+\frac{1}{\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \left(\int_0^{\rho} e^{\rho}\tau\left(\Phi_2(\tau)-\Phi_1(\tau)\right) d\tau\right) d\rho \Bigg| \\ &+\frac{1}{\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \left(\int_0^{1} \rho\tau\left(\Phi_2(\tau)-\Phi_1(\tau)\right) d\tau\right) d\rho \Bigg| \\ \le&\left| \frac{1}{5\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \rho^2 e^{\rho}\left(\Phi_2(\rho)-\Phi_1(\rho)\right) d\rho \right| \\ +&\left| \frac{1}{\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \left(\int_0^{\rho} e^{\rho}\tau\left(\Phi_2(\tau)-\Phi_1(\tau)\right) d\tau\right) d\rho \right| \\ +&\left| \frac{1}{\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \left(\int_0^{1} \rho\tau\left(\Phi_2(\tau)-\Phi_1(\tau)\right) d\tau\right) d\rho \right| \\ \le&\frac{1}{5\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \rho^2 e^{\rho}\left|\Phi_2(\rho)-\Phi_1(\rho)\right| d\rho \\ +&\frac{1}{\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \left(\int_0^{\rho} e^{\rho}\tau\left|\Phi_2(\tau)-\Phi_1(\tau)\right| d\tau\right) d\rho \\ +&\frac{1}{\Gamma\left(\frac{3}{4}\right)} \int_0^{\omega} (\omega-\rho)^{\frac{-1}{4}} \left(\int_0^{1} \rho\tau\left|\Phi_2(\tau)-\Phi_1(\tau)\right| d\tau\right) d\rho \end{aligned}$$

$$\leq \frac{\|\Phi_2-\Phi_1\|_\infty}{5\Gamma\left(\frac{3}{4}\right)(n)!}\int_0^\omega(\omega-\rho)^{-\frac{1}{4}}\rho^{n+2}d\rho+\frac{\|\Phi_2-\Phi_1\|_\infty}{2\Gamma\left(\frac{3}{4}\right)(n)!}\int_0^\omega(\omega-\rho)^{-\frac{1}{4}}\rho d\rho$$
$$+\frac{\|\Phi_2-\Phi_1\|_\infty}{2\Gamma\left(\frac{3}{4}\right)(n)!}\int_0^\omega(\omega-\rho)^{-\frac{1}{4}}\rho^3 d\rho$$
$$\leq\left(\frac{\Gamma(4)\omega^{3.75}}{5\Gamma(4.75)}+\frac{\Gamma(4)\omega^{3.75}}{2\Gamma(4.75)}+\frac{\Gamma(4)\omega^{3.75}}{2\Gamma(4.75)}\right)\|\Phi_2-\Phi_1\|_\infty .$$

Thus,

$$\|T\Phi_2-T\Phi_1\|_\infty\leq(0.54236)\|\Phi_2-\Phi_1\|_\infty .$$

Since $0.54236<1$, we say that the problem satisfies the condition of Theorem 3.2.

Example 2. Consider the fractional differential equation of the form

$$aD^2\Phi(\omega)+b(\omega)D^{\beta_1}\Phi(\omega)+c(\omega)D\Phi(\omega)+e(\omega)D^{\beta_2}\Phi(\omega)$$
$$=f(\omega)-k(\omega)\Phi(\omega),$$

$$\Phi(0)=2,\ \Phi'(0)=0,$$

where $a=1, b(\omega)=\omega^{\frac{1}{2}}, c(\omega)=\omega^{\frac{1}{3}}, e(\omega)=\omega^{\frac{1}{4}}, k(\omega)=\omega^{\frac{1}{5}}, f(\omega)=-a-\frac{b(\omega)}{\Gamma(3-\gamma_1)}\omega^{2-\gamma_1}-c(\omega)\omega-\frac{e(\omega)}{\Gamma(3-\gamma_2)}\omega^{2-\gamma_2}+k(\omega)\left(2-\frac{\omega^2}{2}\right), \gamma_2=0.333$ and $\gamma_1=1.234$.

The exact solution is $\Phi(\omega)=2-\frac{\omega^2}{2}$.

From Theorem 3.2, we have

$$|(T\Phi_2)(\omega)-(T\Phi_1)(\omega)|$$
$$=\left|-\frac{1}{\Gamma(2)\Gamma(1.766)}\int_0^\omega(\omega-\rho)\rho^{1.266}(\Phi_2(\rho)-\Phi_1(\rho))\,d\rho\right.$$
$$-\frac{1}{\Gamma(2)}\int_0^\omega(\omega-\rho)\rho^{1.333}(\Phi_2(\rho)-\Phi_1(\rho))\,d\rho$$
$$-\frac{1}{\Gamma(2)\Gamma(1.667)}\int_0^\omega(\omega-\rho)\rho^{0.917}(\Phi_2(\rho)-\Phi_1(\rho))\,d\rho$$
$$\left.-\frac{1}{\Gamma(2)}\int_0^\omega(\omega-\rho)\rho^{0.2}(\Phi_2(\rho)-\Phi_1(\rho))\,d\rho\right|$$

$$\leq \frac{1}{\Gamma(2)\Gamma(1.766)}\int_0^{\omega}(\omega-\rho)\rho^{1.266}\left|\Phi_2(\rho)-\Phi_1(\rho)\right|d\rho$$
$$+\frac{1}{\Gamma(2)}\int_0^{\omega}(\omega-\rho)\rho^{1.333}\left|\Phi_2(\rho)-\Phi_1(\rho)\right|d\rho$$
$$+\frac{1}{\Gamma(2)\Gamma(1.667)}\int_0^{\omega}(\omega-\rho)\rho^{0.917}\left|\Phi_2(\rho)-\Phi_1(\rho)\right|d\rho$$
$$+\frac{1}{\Gamma(2)}\int_0^{\omega}(\omega-\rho)\rho^{0.2}\left|\Phi_2(\rho)-\Phi_1(\rho)\right|d\rho$$
$$\leq \frac{\left\|\Phi_2-\Phi_1\right\|_\infty}{\Gamma(2)\Gamma(1.766)}\int_0^{\omega}(\omega-\rho)\rho^{1.266}d\rho$$
$$+\frac{\left\|\Phi_2-\Phi_1\right\|_\infty}{\Gamma(2)}\int_0^{\omega}(\omega-\rho)\rho^{1.333}d\rho$$
$$+\frac{\left\|\Phi_2-\Phi_1\right\|_\infty}{\Gamma(2)\Gamma(1.667)}\int_0^{\omega}(\omega-\rho)\rho^{0.917}d\rho$$
$$+\frac{\left\|\Phi_2-\Phi_1\right\|_\infty}{\Gamma(2)}\int_0^{\omega}(\omega-\rho)\rho^{0.2}d\rho.$$

By property (iii) we have

$$\left\|T\Phi_2-T\Phi_1\right\|_\infty$$
$$\leq \Big(\frac{\Gamma(2.266)}{\Gamma(1.766)\Gamma(4.266)}+\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{13}{3}\right)}$$
$$+\frac{\Gamma(1.917)}{\Gamma(1.667)\Gamma(3.917)}+\frac{\Gamma\left(\frac{6}{5}\right)}{\Gamma\left(\frac{16}{5}\right)}\Big)\left\|\Phi_2-\Phi_1\right\|_\infty.$$

Thus,

$$\left\|T\Phi_2-T\Phi_1\right\|_\infty \leq (0.85187)\left\|\Phi_2-\Phi_1\right\|_\infty.$$

Since $0.85187 < 1$, we say that the problem satisfies the condition of Theorem 3.2.

5. **Conclusions**

This study presents the use of the Riemann-Liouville integral of fractional order to the transformation of a new class of multi-term fractional-order Volterra-Fredholm models to its equivalent integral form. The uniqueness of the solution to the multiterm fractional order Volterra-Fredholm integro-differential equation was established by applying the

Banach contraction principle. Additionally, the Cauchy convergence criteria were examined and used to support the convergence study. In order to illustrate the problem's applicability and solvability, instances proving that the condition is satisfied were provided. The outcomes met the uniqueness of solution theorem's criteria with complete harmony.

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