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REGULARIZED ITERATIVE FDEM: A NUMERICAL APPROACH FOR NONLINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we propose a novel numerical technique - Regularized Iterative Fractional Differential Equation Method (FDEM) - for solving nonlinear fractional differential equations (FDEs) involving the Caputo derivative. The method begins by reformulating the fractional differential equation as a Volterra integral equation and addresses the weak singularity in the kernel by a regularization strategy that decomposes the nonlinear term. This decomposition enables a stable and accurate numerical integration using the composite trapezoidal rule. To handle the nonlinearity, a fixed-point iteration is employed at each time step. The resulting algorithm is simple to implement, computationally efficient with $O(m^2)$ complexity, and adaptable to various types of nonlinear FDEs. The method's stability, accuracy, and flexibility make it suitable for practical applications, including systems of fractional equations and higher-dimensional problems. For validating the robustness and effectiveness of the established approach, numerous numerical problems are addressed.

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Key Words and Phrases: iterative method, fractional differential equation, numerical solution

1. **Introduction**

In more recent academic discourse, fractional differential equations (FDEs) have arisen as very vital tools, especially due to their ability to model hereditary dynamics and encapsulate memory in engineering, biological, and diverse physical contexts [1, 2]. A noteworthy implementation of FDEs is the Caputo operator, which turned out to be notable in the modeling of numerous complicated real-world phenomena, see [3, 4, 5, 6, 7, 8, 9, 10, 11]. Such an operator enables the integration of conventional initial conditions, accordingly adjusting properly with initial value problems, an important property that improves its pertinence.

However, there are challenges that beset the numerical resolution of nonlinear FDEs. The most important of these challenges are the properties of the fractional derivatives inherent nonlocality and the complexities presented by weakly singular kernels within their integrals [16, 17]. The presence of nonlinearity aggravates such matters, as classical discretization schemes consistently meet with challenges connected to convergence and stability [18].

To alleviate these complications, we present a new numerical scheme called the Regularized Iterative Finite Difference Element Method (FDEM). Such a novel approach harmonizes regularization methods with an effective iterative discretization technique. The primary notion spins around examining the nonlinear term, enabling the analytical handling of the kernel's singular component. Such an analytical scheme allows the implementation of classical quadrature approaches without the complications of numerical instability frequently related to singularities [19]. In addition, a mechanism of the fixed-point iteration is incorporated to consistently resolve the consequent nonlinear system at every step of the temporal discretization [20].

This work introduces a novel approach for solving nonlinear fractional differential equations involving the Caputo derivative using the Regularized Iterative Finite Difference Element Method (FDEM). The proposed technique reformulates the original problem into a Volterra integral equation and overcomes the challenges of weakly singular kernels through a regularization strategy. Combined with a fixed-point iteration and composite trapezoidal discretization, this method achieves accurate and stable results. The approach is computationally efficient, adaptable to systems of equations, and extendable to higher-dimensional fractional models. The effectiveness of the method is validated through several nonlinear examples with known analytical solutions, demonstrating its robustness and reliability for a wide class of fractional problems.

The execution of the proposed algorithm is straightforward, yielding accurate results while maintaining computational efficiency. The method is flexible and may be extended to systems of nonlinear fractional differential equations. Its effectiveness is confirmed through a series of numerical examples involving nonlinear problems of fractional order. The accuracy of the approach is demonstrated by comparing the obtained numerical results with known analytical solutions, confirming the robustness and reliability of the proposed scheme for solving a broad class of Caputo-type fractional models.

The structure of the remainder of this exposition is as follows: Section 2 delineates the mathematical groundwork of the problem and its conversion into an integral equation format. Section 3 elaborates on the regularization process for the weakly singular kernel. Section 4 provides a comprehensive overview of the numerical discretization and iterative methodology employed. Subsequently, Section 5 elaborates on the overarching algorithm, culminating in Section 6, which underscores the principal attributes and prospective enhancements of the proposed method.

2. **Preliminaries and fractional calculus background**

In this part, we shortly recollect some crucial properties and definitions in relation to fractional calculus that will be very essential to our investigation. Herein, we intend to concentrate on the Caputo and Riemann–Liouville derivative operators as well as the Riemann–Liouville integral operator [21, 22, 23].

Definition 2.1. Suppose f is a real-valued function defined on the interval $[a, b]$. The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > a, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \text{for } z > 0.$$

This operator satisfies the semigroup property:

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \quad \alpha, \beta > 0. \tag{2}$$

It also holds that $J^0 f(t) = f(t)$.

Definition 2.2. For a function $f \in C^n[a, b]$ and $n-1 < \alpha < n$, the Riemann–Liouville fractional derivative of order α is defined as

$$D_{\mathrm{RL}}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds. \tag{3}$$

Actually, the above definition extends the conventional derivative but demands suitable initial conditions in fractional order, which might not be physically explainable in a lot of applications. However, to go beyond the limits of the Riemann–Liouville derivative operator, the Caputo derivative operator was presented.

DEFINITION 2.3. For $f \in C^n[a,b]$ and $n-1<\alpha<n$, the Caputo derivative of order α is defined as

$$D_{\mathrm{C}}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds. \tag{4}$$

The Caputo operator enables employment of conventional initial conditions:

$$f(a), f'(a), \ldots, f^{(n-1)}(a),$$

which is especially beneficial in modeling many real-world applications.

PROPOSITION 2.1. *If $f \in C^n[a,b]$, then the Caputo derivative can be expressed in terms of the Riemann–Liouville derivative as:*

$$D_{\mathrm{C}}^{\alpha} f(t) = D_{RL}^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right). \tag{5}$$

In fact, this formula demonstrates that the Caputo operator efficiently extracts the polynomial initial terms, getting better and more appropriate for problems with conventional initial conditions.

PROPOSITION 2.2. *Some basic characteristics of the Riemann–Liouville integral operator J^α can be mentioned:*

- *Linearity:* $J^\alpha(af+bg) = aJ^\alpha f + bJ^\alpha g$, *where* a, b *are scalars.*
- J^α *of a power function:*

$$J^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}, \quad \beta > -1.$$

- *Connection to identity operator:* $\lim_{\alpha \to 0^+} J^\alpha f(t) = f(t)$.

3. Numerical formulation

Consider the following nonlinear FDE formulated by means of the Caputo operator:

$$D^\alpha u(t) = f(t, u(t)), \quad t \in [a,b], \quad \alpha \in (n-1, n), \quad n \in \mathbb{N}, \tag{6}$$

subject to the initial conditions

$$u^{(k)}(a) = u_k, \quad k = 0, 1, \ldots, n-1. \tag{7}$$

Herein, $D^\alpha u(t)$ represents the Caputo derivative operator of fractional order α, outlined by

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds, \tag{8}$$

which is especially beneficial in establishing physical models, as it enables dealing with classical initial conditions.

To facilitate numerical treatment, we apply the Riemann-Liouville fractional integral operator and reformulate the differential equation into an equivalent Volterra integral equation:

$$u(t) = \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, u(s))\, ds. \tag{9}$$

This integral form reveals a weakly singular kernel $(t-s)^{\alpha-1}$, which poses numerical difficulties near the lower limit $s = t$. To overcome this, we regularize the kernel by decomposing the integrand as

$$f(s, u(s)) = [f(s, u(s)) - f(t, u(t))] + f(t, u(t)), \tag{10}$$

leading to the decomposition of the integral as

$$\begin{aligned} \int_a^t (t-s)^{\alpha-1} f(s, u(s))\, ds = \int_a^t (t-s)^{\alpha-1} [f(s, u(s)) - f(t, u(t))]\, ds \\ + f(t, u(t)) \frac{(t-a)^\alpha}{\alpha \Gamma(\alpha)}. \end{aligned} \tag{11}$$

The first term now has a regularized integrand, while the second term is an explicit analytical expression.

To numerically approximate the regularized integral, we partition the interval $[a, b]$ into m uniform subintervals of step size $h = \frac{b-a}{m}$, with grid points defined as $t_j = a + jh$, for $j = 0, 1, \ldots, m$. We define

$$g(t, s) = \frac{f(s, u(s)) - f(t, u(t))}{\Gamma(\alpha)(t-s)^{1-\alpha}},$$

so that the regularized integral can be approximated using the composite trapezoidal rule:

$$\begin{aligned}\int_a^{t_r} (t_r - s)^{\alpha-1}[f(s,u(s)) - f(t_r,u(t_r))]\,ds \\ \approx \frac{h}{2}\left[g(t_r,a) + 2\sum_{j=1}^{r-1} g(t_r,t_j)\right].\end{aligned} \tag{12}$$

With this discretization in place, we employ a fixed-point iterative scheme to compute $u(t_r)$ for $r = 1, 2, \ldots, m$. Let the initial guess be $u^{(0)}(t_r) = u(t_{r-1})$. Then the update rule at each iteration ℓ is given by

$$\begin{aligned}u^{(\ell+1)}(t_r) = \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t_r - a)^k + \frac{f(t_r, u^{(\ell)}(t_r))(t_r - a)^\alpha}{\alpha\Gamma(\alpha)} \\ + \frac{h}{2}\left[g(t_r,a) + 2\sum_{j=1}^{r-1} g(t_r,t_j)\right].\end{aligned} \tag{13}$$

The iteration continues until convergence is achieved, which is determined by the condition

$$|u^{(\ell+1)}(t_r) - u^{(\ell)}(t_r)| < \mathrm{tol},$$

where `tol` is a user-specified tolerance.

This formulation forms the core of our proposed method and sets the stage for the full algorithm implementation and subsequent analysis.

4. General algorithm (Pseudocode)

The previous section, which has included a desired numerical formulation, is applied using an effective step-by-step algorithm designed to address nonlinear problems formulated in the Caputo sense. The intended algorithm blends regularization, discretization via the trapezoidal rule, and fixed-point iteration to manage the weak singularity and nonlinearity simultaneously. In the succeeding content, we condense the complete computational approach in the form of a general pseudocode.

- **Input:** $\alpha \in (n-1, n)$, interval $[a, b]$, number of steps m, function $f(t, u)$, initial values $\{u^{(k)}(a)\}_{k=0}^{n-1}$.
- **Output:** Approximate solution $\{u(t_r)\}_{r=0}^{m}$.

Initialize:

- Set $h = (b - a)/m$, $t_r = a + rh$.

- Assign $u(t_0) = u_0$.
- Precompute $C_k = \frac{u^{(k)}(a)}{k!}$ for $k = 0, 1, \ldots, n-1$.

Main Loop:

- For $r = 1$ to m:
  - Compute $t_r = a + rh$.
  - Set $u^{(0)}(t_r) = u(t_{r-1})$.
  - Repeat until convergence:

$$u^{(\ell+1)}(t_r) = \sum_{k=0}^{n-1} C_k (t_r - a)^k + \frac{f(t_r, u^{(\ell)}(t_r))(t_r - a)^\alpha}{\alpha\Gamma(\alpha)} + \frac{h}{2}\left[g(t_r, a) + 2\sum_{j=1}^{r-1} g(t_r, t_j)\right].$$

  - Set $u(t_r) = u^{(\text{final})}(t_r)$.

5. **Numerical examples**

To exhibit the efficacy and capability of the proposed approach, we introduce three demonstrative examples of FDEs. Such examples address numerous nonlinear problems and verify the precision of the yielded analytical solutions.

Example 5.1. ([24]) To demonstrate the effectiveness of the proposed Regularized Iterative FDEM method, we consider the nonlinear fractional differential equation:

$$D^\alpha u(t) + u(t) = te^{-t},\ 1 < \alpha < 2,\ t \in [0, 20],\ u(0) = 0,\ u'(0) = 0, \quad (14)$$

where $D^{1.5}$ denotes the Caputo fractional derivative of order $\alpha = 1.5$. This problem is challenging due to the nonlinear term and the fractional-order derivative.

The function $f(t, u(t)) = -u(t) + te^{-t}$ is substituted into the integral formulation and handled using the regularized kernel approach described earlier. We divide the interval $[0, 20]$ into $m = 100$ equal subintervals, and use a fixed-point iteration with a maximum of 100 iterations per grid point.

The exact solution is computed numerically using a convolution with the Mittag-Leffler function:

$$u(t) = \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}(-(t - s)^\alpha) \cdot se^{-s}\, ds,$$

where $E_{\alpha,\alpha}$ is the two-parameter Mittag-Leffler function. This solution is evaluated using numerical integration.

The results in Table 1 compare the numerical approximation and the reference solution at selected points. The absolute error is also reported.

TABLE 1. Comparison between numerical and exact solutions for Example 1 with $\alpha = 1.5$, $m = 100$.

t	$u_{\text{num}}(t)$	$u_{\text{exact}}(t)$	$\lvert u_{\text{num}} - u_{\text{exact}}\rvert$
2.0	0.357863	0.358557	6.94×10^{-4}
4.0	0.201659	0.201226	4.34×10^{-4}
6.0	-0.034101	-0.034040	6.15×10^{-5}
8.0	-0.023594	-0.023606	1.18×10^{-5}
10.0	0.004042	0.004012	3.06×10^{-5}
12.0	0.000419	0.000428	8.23×10^{-6}
14.0	-0.002005	-0.002003	2.22×10^{-6}
16.0	-0.000724	-0.000729	4.08×10^{-6}
18.0	-0.000263	-0.000265	1.20×10^{-6}
20.0	-0.000319	-0.000319	1.53×10^{-7}

The results indicate that the numerical method provides an accurate approximation of the exact solution, with the error decreasing as t increases. This validates the accuracy and efficiency of the Regularized Iterative FDEM for fractional nonlinear problems.

EXAMPLE 5.2. ([24]) In this example, we consider a nonlinear fractional differential equation involving the Caputo derivative of order $\alpha = 0.9$ over the interval $[0, 1]$. The equation is defined as:

$$D^{0.9}u(t) = -u(t) + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)}, \quad u(0) = 0, \quad u'(0) = 0. \tag{15}$$

The nonlinear term $f(t, u(t)) = -u(t) + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)}$ ensures the existence of a known exact solution:

$$u(t) = t^4 E_{\alpha,5}(-t^{\alpha}),$$

where $E_{\alpha,\beta}(\cdot)$ is the Mittag-Leffler function.

The solution is approximated using the proposed Regularized Iterative FDEM method. The interval $[0, 1]$ is divided into $m = 100$ subintervals, and the fixed-point iteration is applied at each grid point. The numerical results are compared with the exact solution, and the absolute error is computed.

Table 2 presents the computed values of $u(t)$, the exact solution, and the corresponding absolute errors at selected points.

Table 3 presents the computed values of the error at different values of α.

TABLE 2. Comparison between numerical and exact solutions for Example 2 with $\alpha = 0.9$, $m = 100$.

t	$u_{\text{num}}(t)$	$u_{\text{exact}}(t)$	$\lvert u_{\text{num}} - u_{\text{exact}}\rvert$
0.0	0.000000	0.000000	0.000000
0.1	4.096×10^{-6}	4.045×10^{-6}	5.04×10^{-8}
0.2	6.332×10^{-5}	6.313×10^{-5}	2.00×10^{-7}
0.3	3.126×10^{-4}	3.122×10^{-4}	4.36×10^{-7}
0.4	9.658×10^{-4}	9.650×10^{-4}	7.43×10^{-7}
0.5	2.307×10^{-3}	2.306×10^{-3}	1.11×10^{-6}
0.6	4.687×10^{-3}	4.685×10^{-3}	1.53×10^{-6}
0.7	8.511×10^{-3}	8.509×10^{-3}	1.98×10^{-6}
0.8	1.424×10^{-2}	1.424×10^{-2}	2.47×10^{-6}
0.9	2.239×10^{-2}	2.238×10^{-2}	2.98×10^{-6}
1.0	3.350×10^{-2}	3.350×10^{-2}	3.51×10^{-6}

TABLE 3. Error values at different values of α for Example2.

t	**At** $\alpha = 0.5$	**At** $\alpha = 0.75$	**At** $\alpha = 0.9$	**At** $\alpha = 1$
0.0	0	0	0	0
0.1	6.28×10^{-8}	5.32×10^{-8}	4.06×10^{-8}	3.17×10^{-8}
0.2	3.58×10^{-7}	2.59×10^{-7}	1.80×10^{-7}	1.32×10^{-7}
0.3	9.41×10^{-7}	6.16×10^{-7}	4.08×10^{-7}	2.89×10^{-7}
0.4	1.83×10^{-6}	1.11×10^{-6}	7.09×10^{-7}	4.90×10^{-7}
0.5	3.04×10^{-6}	1.73×10^{-6}	1.07×10^{-6}	7.26×10^{-7}
0.6	4.57×10^{-6}	2.47×10^{-6}	1.48×10^{-6}	9.88×10^{-7}
0.7	6.42×10^{-6}	3.30×10^{-6}	1.93×10^{-6}	1.26×10^{-6}
0.8	8.60×10^{-6}	4.23×10^{-6}	2.42×10^{-6}	1.56×10^{-6}
0.9	1.11×10^{-5}	5.24×10^{-6}	2.93×10^{-6}	1.86×10^{-6}
1.0	1.39×10^{-5}	6.32×10^{-6}	3.45×10^{-6}	2.17×10^{-6}

The results confirm the high accuracy of the method, with errors remaining small across the entire interval. This validates the applicability of the Regularized Iterative FDEM to fractional models with known analytical solutions, further confirming its reliability for solving nonlinear FDEs.

EXAMPLE 5.3. ([25]) In this example, we apply the Regularized Iterative FDEM to solve the nonlinear fractional Riccati differential equation:

$$D^1 u(t) = u(t) - u^2(t), \quad u(0) = 0.5. \tag{16}$$

This is a classic logistic-type growth model, where the fractional order is $\alpha = 1$, corresponding to the classical first-order derivative. The exact solution for this equation is known and given by

$$u(t) = \frac{1}{1 + e^{-t}}, \tag{17}$$

which represents a sigmoid function with asymptotic behavior

$$u(t) \to 1 \text{ as } t \to \infty.$$

We solve this equation numerically on the interval $[0, 20]$, using $m = 100$ subintervals. The right-hand side function is $f(t, u(t)) = u(t) - u^2(t)$. We apply a fixed-point iteration with up to 50 iterations per grid point, using the regularization and discretization strategies described earlier.

The computed results are presented in Table 4, showing the approximate solution, exact solution, absolute error, and relative error at selected points.

The numerical results closely match the exact sigmoid solution, with both absolute and relative errors remaining extremely small. This confirms the robustness of the method even for long time intervals and nonlinearity-dominated dynamics.

6. **Conclusion**

This paper introduces an innovative numerical approach known as Regularized Iterative FDEM, designed to address nonlinear fractional differential equations that incorporate the Caputo derivative. The proposed method involves transforming the original equation into a corresponding Volterra integral format while regularizing the weakly singular kernel. This transformation is pivotal in mitigating the complications associated with the nonlocal and singular characteristics inherent in fractional

Table 4. Comparison between numerical and exact solutions for Example 3 with $\alpha = 1$, $m = 100$.

t	$u_{\text{num}}(t)$	$u_{\text{exact}}(t)$	**Error**	**Relative Error**
0.0	0.500000	0.500000	0.000000	0.000000
2.0	0.880697	0.880797	1.00×10^{-4}	1.14×10^{-4}
4.0	0.982079	0.982014	6.54×10^{-5}	6.66×10^{-5}
6.0	0.997552	0.997527	2.48×10^{-5}	2.49×10^{-5}
8.0	0.999670	0.999665	5.57×10^{-6}	5.58×10^{-6}
10.0	0.999956	0.999955	1.05×10^{-6}	1.05×10^{-6}
12.0	0.999994	0.999994	1.83×10^{-7}	1.83×10^{-7}
14.0	0.999999	0.999999	3.01×10^{-8}	3.01×10^{-8}
16.0	0.99999989	0.99999989	4.80×10^{-9}	4.80×10^{-9}
18.0	0.99999999	0.99999998	7.47×10^{-10}	7.47×10^{-10}
20.0	1.00000000	1.00000000	1.14×10^{-10}	1.14×10^{-10}

Table 5. Approximate solution values for different α at selected points $t \in [0, 20]$ using $m = 100$.

t	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$	$\alpha = 1$
0	0.5	0.5	0.5	0.5
2	0.880697	0.872192	0.857612	0.827737
4	0.982079	0.958977	0.926419	0.872337
6	0.997552	0.980594	0.951659	0.894632
8	0.999670	0.987852	0.963901	0.908503
10	0.999956	0.991131	0.970993	0.918155
12	0.999994	0.992964	0.975603	0.925352
14	0.999999	0.994139	0.978844	0.930974
16	1.000000	0.994962	0.981252	0.935519
18	1.000000	0.995572	0.983117	0.939289
20	1.000000	0.996043	0.984606	0.942479

operators. By employing a decomposition technique along with a fixed-point iterative method, the framework facilitates a stable and precise computation of the numerical solution. Additionally, the implementation of the composite trapezoidal rule enhances the method's usability, achieving second-order accuracy in the approximation of the regularized integral. Noteworthy advantages of this approach lie in its adaptability to various nonlinearities, its applicability to both individual scalar equations and multi-variable systems, and its potential for expansion to encompass higher-order or multidimensional fractional models. Future inquiry will focus on the theoretical underpinnings related to stability and convergence, the adaptation to variable-order and multi-term fractional systems, and the practical application of the method in domains such as physics, biology, and engineering. Research directions include theoretical analysis of stability and convergence, extension to variable-order and multi-term fractional systems, and application to real-world problems in physics, biology, and engineering.

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