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ON ISOLATED POINTS OF THE THIN SUPERKERNEL OF COMPACT SPACES

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Abstract

In this work, we study some properties of the set of isolated points of the thin superkernel of compact spaces. We prove that for a T_1-space, we have that $I(\lambda X) \subset \lambda^* X$, i.e. each isolated point in λX must be a thin maximal linked system. Also prove that for an infinite compact space X with $I(X) = \emptyset$, then $I(\lambda X) = \emptyset$, i.e. if X is a pointless space, then its superextension λX is also pointless.

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1. **Introduction and preliminaries**

Recently, the spaces of linked systems in topology have been studied by many authors (see [1], [3], [7]). They considered several cardinal properties and studied the relation between a space X satisfying such a property and of space of the linked systems of X.

The notion of linked systems wins an essential part in the branches of theory functors and theory cardinal invariants (see for example [2], [6], [8]). Several properties of theory functors and theory cardinal invariants have been studied in the view of spaces of the linked systems and their cardinal invariants.

Simultaneously, the importance of this research issue; the meaning of linked systems, led to define new types of linked systems; namely compact maximal linked systems and compact complete linked systems for compact spaces; $\mathcal{N}-$compact nucleus, $\mathcal{N}-$subtle nucleus, $\mathcal{N}_\tau^\varphi-$ nucleus of a topological space X (see for example [4], [5], [8]).

On the other hand, since the spaces of maximal and complete linked systems have their own significant role in the field of Topology, it is absolutely natural to have new related realms of spaces. Especially, in [7], the authors studies the notion of space of complete linked systems as a generalization of the space of maximal linked systems. Note that any maximal linked system is complete, but the converse is not always true. In these works we can also find an initial study of cardinal and hereditary cardinal invariants of the space of compact complete linked systems.

By establishing these results, we contribute to a broader understanding of the kernels of topological spaces in topology and their implications for isolated points. This research opens avenues for further exploration of functorial interactions in generalized topological settings.

A system $\xi = \{F_\alpha : \alpha \in A\}$ of closed subsets of a space X is called linked, if any two elements of ξ intersect. Any linked system can be upgraded to a maximum linked system (MLS). But such upgrade, as a rule, is not one valued. A linked system of space is MLS, if and only if it possesses the following completeness property [3]: "If a closed set $A \subset X$ intersects with every element of ξ, then $A \in \xi$". We denote λX as the set of all MLS of the space X. For the open set $U \subset X$ we consider $O(U) = \{\xi \in \lambda X : \text{there exists } F \in \xi \text{ such that } F \subset U\}$.

The family of sets of the form $O(U)$ covers the set λX, i.e. $O(X) = \lambda X$. So, it forms an open prebase of the topology on λX. The set λX, equipped with this topology, is called as the superextension of the space

X. Let X be topological space and λX be its superextension. MLS $\xi \in \lambda X$ is called thin, if it contains at least one finite element, and is denoted by TMLS. The space $\lambda^* X = \{\xi \in \lambda X : \ \xi \ is \ TMLS\}$ we call as thin super kernel (or thin superextension) of the topological space X [4].

For some undefined or related concepts, we refer the reader to [3, 4].

2. **Main results**

Denote by $I(\lambda X)$ the set of all isolated points of the superextension λX. Let us see what a connection exists between $I(\lambda X)$ and $\lambda^* X$.

THEOREM 2.1. *Let X be a T_1-infinity space, then we have that $I(\lambda X) \subset \lambda^* X$, i.e. each isolated point in λX must be a TMLS.*

P r o o f. Let $\xi \in I(\lambda X)$ and show that $\xi \in \lambda^* X$. Since ξ is an isolated point in λX, there exists a minimal neighborhood $O = O(U_1, U_2, \ldots, U_n)$ points ξ in λX, i.e. $O = \{\xi\}$. Let $S(O) = \{V_1, V_2, \ldots, V_s\}$ - pairwise trace of the element O in X. In each set $V_i \in S(O)$ we fix one point x_i, then we get the set $\sigma = \{x_1, x_2, \ldots, x_s\}$. We set $\Phi_i = \{x_j \in \sigma : \ x_j \in U_i\}$, where $i = 1, 2, \ldots, n$. Obviously, the system $\mu = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}$ is linked system of sets closed in X. Since $\Phi_i \subset U_i$, where $i = 1, 2, \ldots, n$, then $\mu^+ \subset O = \{\xi\} \Rightarrow \mu \subset \xi$. Therefore, ξ is a TMLS, i.e. $\xi \in \lambda^* X$. Since ξ is arbitrary in $I(\lambda X)$, we have $I(\lambda X) \subset \lambda^* X$. Theorem 2.1 is proved. □

COROLLARY 2.1. *Let X be an infinite compact space and $I(X) = \emptyset$, then $I(\lambda X) = \emptyset$, i.e. if X is a pointless space, then its superextension λX is also pointless.*

It is known that if X is compact, then $q(X) \leq \omega$ ($q(X)$ is Hewitt-Nachbin number of the space X). So, if X is a normal space, then always $q(X) \leq \omega$.

For the Hewitt-Nachbin number of the space $\lambda^* X$ we have the following result.

THEOREM 2.2. *If X is compact, then $q(\lambda^* X) \leq d(X)$.*

P r o o f. It is known that $q(X) = t_R(C_p(X)) = t_\theta(C_p(X))$ and $t_\theta \leq t_c(X) \leq d(X)$ for any Tikhonoff space X. So, we have $q(\lambda^* X) =$

$t_R(C_p(\lambda^* X)) = t_\theta(C_p(\lambda^* X)) \leq t_c(\lambda^* X) \leq d(C_p(\lambda^* X)) \leq d(\lambda^* X) = d(X)$.

Theorem 2.2 is proved. □

Corollary 2.2. *If X is a separable compact, then $q(\lambda^* X) \leq \omega$.*

Corollary 2.3. *If X is a separable compact, then the space $\lambda^* X$ is a $Q_\omega-$space.*

Now consider the following question: When is an thin superkernel $\lambda^* X$ $\tau-$placed in λX?

Theorem 2.3. *If $q(\lambda^* X) \leq \tau$ and $m(\lambda X) \leq \tau$, then $\lambda^* X$ is $\tau-$placed in λX.*

P r o o f. A. Ch. Chigogidze showed that if $X \subset Y$, $q(X) \leq \tau$, $m(Y) \leq \tau$ and X is everywhere dense in Y, then X is $\tau-$placed in Y. Since $\lambda^* X$ is everywhere dense in λX, $q(\lambda^* X) \leq \tau$ and $m(\lambda X) \leq \tau$, we get that $\lambda^* X$ is $\tau-$placed in λX.

Theorem 2.3 is proved. □

Corollary 2.4. *If $m(\lambda X) \leq \tau = d(X)$, then the thin superkernel $\lambda^* X$ is $\tau-$placed in λX.*

Corollary 2.5. *If λX is Moscow space and X is separable, then the thin superkernel $\lambda^* X$ is $\omega-$placed in λX.*

Theorem 2.4. *A thin superkernel of any infinite T_1-space X is everywhere dense in λX, i.e. $[\lambda^* X]_{\lambda X} = \lambda X$.*

P r o o f. Let $O = O\,(U_1, U_2, \ldots, U_n)$ be an arbitrary non-empty open basic element in λX. Consider the pairwise trace of an element $O = O\,(U_1, U_2, \ldots, U_n)$ in X, i.e. consider the system $S\,(O) = \{V_1, V_2, \ldots, V_l\}$ of all possible pairwise intersections of the class $K\,(O) = \{U_1, U_2, \ldots, U_n\}$, where $K\,(O)$ is the skeleton of an element $O = O\,(U_1, U_2, \ldots, U_n)$ in X.

Let in every set $V_i \in S\,(O)$ we fix one point x_i $(i = 1, 2, \ldots, n)$. Then we obtain the set $\sigma = \{x_1, x_2, \ldots, x_n\}$. We consider $\Phi_i = \{x_j \in \sigma : x_j \in U_i\}$, where $i, j = 1, 2, \ldots, n$. It is easy to check that the system $\mu = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}$ is linked system of closed sets in X and $\Phi_i \subset U_i$, $i =$

1, 2, ..., n. Consequently, we have that $\mu^+ = \{\xi \in \lambda X : \mu \subset \xi\} \subset O = O(U_1, U_2, \ldots, U_n)$. Since $\mu^+ \subset \lambda^* X$, then we obtain that $\lambda^* X \cap \neq \emptyset$. By virtue of arbitrariness of open basic element $O = O(U_1, U_2, \ldots, U_n)$ in λX, we have that $\lambda^* X$ is everywhere dense in λX, i.e. $[\lambda^* X]_{\lambda X} = \lambda X$. Theorem 2.4 is proved. □

Corollary 2.6. *Let X be an infinite T_1-space, then the following properties are equivalent:*

1) X is separable;
2) $\lambda^* X$ is separable.

Corollary 2.7. *Let X be an infinite compact space, then the following inequalities are true:*

1) $ld(X) \neq ld(\lambda^* X)$;
2) $wd(X) \neq wd(\lambda^* X)$;
3) $lwd(X) \neq lwd(\lambda^* X)$.

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