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PARTIAL ORDER IN MODULE OVER A MATRIX NEARRING

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Abstract

Let $M_n(N)$ be a matrix nearring over the nearring N with identity and let N^n be the direct sum of n-copies of the group $(N, +)$. We introduce a partial order in the $M_n(N)$-group N^n corresponding to the partial order in N-group (over itself). We define a positive cone in $M_n(N)$-group N^n and obtain its characterization. For a convex ideal of ${}_NN$, the corresponding ideal in $M_n(N)$-group N^n is described; and conversely, if $\mathcal{I}$ is a convex ideal in $M_n(N)$-group N^n, then the ideal $\mathcal{I}_{**}$ is convex in N (over itself). This establishes the one-one correspondence between the convex ideals of the p.o. N-group ${}_NN$ and those of p.o. $M_n(N)$-group N^n.

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1. **Introduction**

The notion of partial order in algebraic systems such as in groups, rings and modules are known [6]. However, the notion of partially ordered nearrings (in short, p.o. nearring) was defined by Pilz [8, 9]. Some developments in the ideal theory of parially ordered nearrings and lattice ordered nearrings were found in [13]. The purpose of this paper is to introduce and study the matrix nearrings over partial order nearrings. Matrix nearrings over arbitrary nearrings were introduced in Meldrum & Van der Walt [10], where several results about the correspondence between the two-sided ideals in the base nearring N and those in the matrix nearring $M_n(N)$ were proved. Later, remarkable developments in matrix nearrings over arbitrary nearrings were due to Meldrum and Meyer [11], Meyer [12]. Corresponding to an ideal in the base nearring N, Meldrum and Meyer [11] have shown that an arbitrary large lattice of ideals in the matrix nearring, under some suitable assumptions. Recently, several authors [1, 3, 14] were extensively studied ideal theory in matrix nearrings. In Bhavanari and Kuncham [4], the uniform and essential ideals of module over a matrix nearring were introduced and obtained a characterization theorem for finite Goldie dimension. We refer to Meldrum and Van der Walt [10], for comprehensive literature on matrix nearrings.

In this paper we introduce the partial order in $M_n(N)$-group N^n based on the partial order defined in $M_n(N)$ as in Tapatee et al. [17]. We define a positive cone in the $M_n(N)$-group N^n and prove a characterization theorem. For a convex ideal of N over N, we establish the corresponding ideal in $M_n(N)$-group N^n; and conversely, if $\mathcal{I}$ is a convex ideal in $M_n(N)$-group N^n, then the ideal $\mathcal{I}_{**}$ is convex in N (over itself). This establishes the one-one correspondence between the convex ideal of p.o. N-group over N and those of p.o. $M_n(N)$-group N^n.

An algebraic structure $(N, +, \cdot)$ is called a (right) nearring if: (i) $(N, +)$ is a group (not necessarily abelian); (ii) $(N, \cdot)$ is a semigroup; and (iii) $(a + b)c = ac + bc$ for all $a, b, c \in N$. Obviously, if $(N, +, .)$ is a right nearring, then $0a = 0$ and $(-a)b = -ab$, for all $a,\ b \in N$, but in general $a0 \neq 0$ for some $a \in N$. If $a0 = 0$, for all $a \in N$, then N is called zero-symmetric, and we denote as $N = N_0$. If $aa' = a$, or $a0 = a$, for all $a \in N$, then N is called a constant nearring, we denote as $N = N_c$.

We use $\Longleftrightarrow$ for 'íf and only if'.

2. Partial order in $M_n(N)$-group N^n

According to Meldrum & Van der Walt [10]: For a zero-symmetric right nearring N with identity 1, let N^n denote the direct sum of n copies of $(N, +)$. The elements of N^n are thought of as column vectors and written as $\langle r_1, \cdots, r_n\rangle$. The symbols i_j and π_j will denote the i^{th} coordinate injection and j^{th} coordinate projection functions respectively. The nearring of $n\times n$-matrices over N, denoted by $M_n(N)$, is defined to be the subnearring of $M(N^n)$, generated by the set of functions $\{f^r_{ij} : N^n \to N^n \mid r \in N, 1 \leq i, j \leq n\}$ where $f^r_{ij}\langle r_1, r_2, \cdots, r_n\rangle := \langle s_1, s_2, \cdots, s_n\rangle$ with $s_i = rr_j$ and $s_k = 0$ if $k \neq i$. The elements of $M_n(N)$ will be referred to as $n \times n$-matrices over N. The zero matrix in $M_n(N)$ is denoted by $\mathbf{0} = 0_{M_n(N)}$, and zero element in N^n is denoted by $\bar{0}$.

Any matrix A can be represented as an expression involving only the f^r_{ij}. The length of such an expression is the number of f^r_{ij} therein. The weight $w(A)$ of A is the length of an expression of minimal length for A. Clearly, if A is represented by an expression of length $w(A) \geq 2$, then from this expression we can find representations for A as either $A = B + C$ or $A = BC$, where $w(B), w(C)<w(A)$.

Notation 2.1. We denote the partial order in N, $M_n(N)$ and N^n as $\leq$, $\leq_n$ and $\leq_{N^n}$, respectively.

Definition 2.1. Let N be a partially ordered nearring with 1.

(1) For any $A, B \in M_n(N)$, we define
$A \leq_n B$ if and only if $\pi_i(A\rho) \leq \pi_i(B\rho)$, for all $\rho \in (P_N)^n$, $1 \leq i \leq n$.

(2) $M_n(N)$ is said to be a p.o. matrix nearring if $\leq_n$ defined in (1) is a partial order and satisfy the monotone properties of addition and multiplication in $M_n(N)$. That is,
 (a) $A \geq_n \mathbf{0}$ and $B \geq_n \mathbf{0}$ implies $A + B \geq_n \mathbf{0}$;
 (b) $A \leq_n B$ and $\mathbf{0} \leq_n C$ implies $AC \leq_n BC$, and $CA \leq_n CB$.

Definition 2.2. Let $M_n(N)$ be a p.o. matrix nearring and N^n be a p.o. group. Define $\leq_{N^n}$ on $M_n(N)$-group N^n by

$$\rho_1 \leq_{N^n} \rho_2 \text{ if and only if } \pi_i(A\rho_1) \leq \pi_i(A\rho_2), \text{ for all } A \in M_n(N),\ 1 \leq i \leq n.$$

N^n is said to be a p.o. $M_n(N)$-group if $\leq_{N^n}$ is a partial order and satisfy the monotone properties of addition and multiplication on N^n. That is,

(1) $\rho_1 \geq_{N^n} \bar{0}$ and $\rho_2 \geq_{N^n} \bar{0}$ implies $\rho_1 + \rho_2 \geq_{N^n} \bar{0}$;
(2) $\rho_1 \leq_{N^n} \rho_2$ and $\mathbf{0} \leq_n B$ implies $B\rho_1 \leq_{N^n} B\rho_2$;
(3) $A \leq_n B$ and $\bar{0} \leq_{N^n} \rho$ implies $A\rho \leq_{N^n} B\rho$,

for all $\rho, \rho_1, \rho_2 \in N^n$, and $A, B \in M_n(N)$.

Definition 2.3. The positive cone of a p.o. $M_n(N)$-group N^n is defined as $\mathcal{P}_{N^n} = \{\rho \mid \rho \geq_{N^n} \bar{0}\}$.

Lemma 2.4. $\mathcal{P}_{N^n}$ satisfies

(1) $\mathcal{P}_{N^n} + \mathcal{P}_{N^n} = \mathcal{P}_{N^n}$,
(2) $\mathcal{P}_{M_n(N)}\mathcal{P}_{N^n} \subseteq \mathcal{P}_{N^n}$,
(3) $\mathcal{P}_{N^n} \cap \mathcal{P}_{N^n} = \{\bar{0}\}$,
(4) $\rho + \mathcal{P}_{N^n} - \rho \subseteq \mathcal{P}_{N^n}$, for all $\rho \in N^n$.

P r o o f. (i) Let $\rho \in \mathcal{P}_{N^n} + \mathcal{P}_{N^n}$. Then $\rho = \rho_1 + \rho_2$, for some $\rho_1, \rho_2 \in \mathcal{P}_{N^n}$. That is, $\pi_i(A\rho_1) \geq 0$ and $\pi_i(A\rho_2) \geq 0$, for all $A \in M_n(N)$. This implies $\pi_i(A\rho_1 + A\rho_2) = \pi_i(A\rho_1) + \pi_i(A\rho_2) \geq 0$. Now, since $\rho_1 \leq_{N^n} \rho_1 + \rho_2$ and $\mathbf{0} \leq_n A$, by monotonicity we have $A\rho_1 \leq_{N^n} A(\rho_1 + \rho_2)$. Similarly, $A\rho_2 \leq_{N^n} A(\rho_1 + \rho_2)$. Then $A\rho_1 + A\rho_2 \leq_{N^n} A(\rho_1 + \rho_2) + A(\rho_1 + \rho_2)$. Hence, $A\rho_1 + A\rho_2 \leq_{N^n} A(\rho_1 + \rho_2)$. Now $0 \leq \pi_i(A\rho_1 + A\rho_2) \leq \pi_i(A(\rho_1 + \rho_2)) = \pi_i(A\rho)$. Therefore, $\pi_i(A\rho) \geq 0$. Hence, $\rho \in \mathcal{P}_{N^n}$, shows that $\mathcal{P}_{N^n} + \mathcal{P}_{N^n} \subseteq \mathcal{P}_{N^n}$. Conversely, suppose that $\rho \in \mathcal{P}_{N^n}$. Let $\rho = \langle a_1, \cdots, a_n \rangle$. Then,

$\rho = \langle a_1, \cdots, a_n \rangle = \langle a_1 + 0, \cdots, a_n + 0 \rangle = \langle a_1, \cdots, a_n \rangle + \langle 0, \cdots, 0 \rangle \in \mathcal{P}_{N^n} + \mathcal{P}_{N^n}$. Therefore, $\mathcal{P}_{N^n} \subseteq \mathcal{P}_{N^n} + \mathcal{P}_{N^n}$. Hence, $\mathcal{P}_{N^n} + \mathcal{P}_{N^n} = \mathcal{P}_{N^n}$.

(ii) Let $A \in \mathcal{P}_{M_n(N)}$ and $\rho \in \mathcal{P}_{N^n}$. Let $B \in \mathcal{P}_{M_n(N)}$. Now, since $\mathbf{0} \leq_n A$, $\mathbf{0} \leq_n B$ in $M_n(N)$, by monotonicity in $M_n(N)$, we get $\mathbf{0} \leq_n BA$. Also, since $\rho \in \mathcal{P}_{N^n}$, by monotonicity in $M_n(N)$-group N^n,we have $(BA)\rho \in \mathcal{P}_{N^n}$. Then, $\pi_i(B(A\rho)) = \pi_i(BA)\rho \geq 0$, for all i, and $B \in \mathcal{P}_{M_n(N)}$. Therefore, $A\rho \in \mathcal{P}_{N^n}$.

(iii) Clearly, $-\mathcal{P}_{N^n} = \{\rho \mid \rho \leq_{N^n} \bar{0}\}$. Then, $\mathcal{P}_{N^n} \cap -\mathcal{P}_{N^n} = \{\bar{0}\}$.

(iv) Let $\rho = \langle x_1, \cdots, x_n \rangle$ and $\rho_1 = \langle a_1, \cdots, a_n \rangle \in \mathcal{P}_{N^n}$. Take $A \in \mathcal{P}_{M_n(N)}$. Then, $\rho + \rho_1 - \rho = \langle x_1 + a_1 - x_1, \cdots, x_n + a_n - x_n \rangle \geq_{N^n} \bar{0}$. Hence, $x_i + a - x_i \in \mathcal{P}_{N^n}$. That is, $\pi_i(A(\rho + \rho_1 - \rho)) \geq 0$. Therefore, $\rho + \rho_1 - \rho \in \mathcal{P}_{N^n}$. □

Proposition 2.5. If N is p.o. N-group, then N^n is a p.o. $M_n(N)$-group N^n.

P r o o f. Suppose N is a p.o. N-group. To show $\leq_{N^n}$ is a partial order relation on $M_n(N)$-group N^n. We have $\pi_i(A\rho) \leq \pi_i(A\rho)$, for all $A \in M_n(N)$, $\rho \in N^n$, and for all i. Hence, $\rho \leq_{N^n} \rho$. Suppose $\rho_1 \leq_{N^n} \rho_2$ and $\rho_2 \leq_{N^n} \rho_1$. Then $\pi_i(A\rho_1) \leq \pi_i(A\rho_2)$ and $\pi_i(A\rho_2) \leq \pi_i(A\rho_1)$, for all $A \in M_n(N)$. Hence, $\pi_i(A\rho_1) = \pi_i(A\rho_2)$, as $\leq$ is p.o. in N. Therefore, $\rho_1 = \rho_2$. To show $\leq_{N^n}$ is transitive, let $\rho_1 \leq_{N^n} \rho_2$ and $\rho_2 \leq_{N^n} \rho_3$. That is, $\pi_i(A\rho_1) \leq \pi_i(A\rho_2)$ and $\pi_i(A\rho_2) \leq \pi_i(A\rho_3)$. Hence, $\pi_i(A\rho_1) \leq \pi_i(A\rho_3)$, as $\leq$ is p.o. in N. Therefore, $\rho_1 \leq_{N^n} \rho_3$.

Now we show monotonicity, that is:

(1) $\rho_1 \leq_{N^n} \rho_2$ and $\mathbf{0} \leq_n B$ implies $B\rho_1 \leq_{N^n} B\rho_2$;
(2) $A \leq_n B$ and $\bar{0} \leq_{N^n} \rho$ implies $A\rho \leq_{N^n} B\rho$.

Let $\rho_1 \leq_{N^n} \rho_2$ in $(N^n, +)$ and $\mathbf{0} \leq_n B$ in $M_n(N)$. Take $\rho_1 = \langle x_1, \cdots, x_n \rangle$ and $\rho_2 = \langle y_1, \cdots, y_n \rangle$. Then $\rho_1 \leq_{N^n} \rho_2$ implies $\pi_i(A\rho_1) \leq \pi_i(A\rho_2)$, for all $A \in M_n(N)$. In particular, $\pi_i(B\rho_1) \leq \pi_i(B\rho_2)$, for all i. This shows that $B\rho_1 \leq_{N^n} B\rho_2$.

We use the induction on weight of B. Let $w(B) = 1$ and $B = f^r_{ij}$, $r \in P_N$. Then for any $x_j \leq y_j$, by monotonicity in N, $rx_j \leq ry_j$. Then, $\langle 0, \cdots, rx_j, \cdots, 0 \rangle \leq_n \langle 0, \cdots, ry_j, \cdots, 0 \rangle$ in $(N^n, +)$. That is, $f^r_{ij}\langle x_1, \cdots, x_n \rangle \leq_n f^r_{ij}\langle y_1, \cdots, y_n \rangle$. Therefore, $B\rho_1 \leq_{N^n} B\rho_2$. We assume that the result is true for $w(B) < n$. Suppose $w(B) = n$. Then $B = C+D$ or $B = CD$.

Case (i): Let $B = C + D$.

Then,

$$\begin{aligned} B\rho_1 &= (C+D)\rho_1 \\ &= C\rho_1 + D\rho_1 \\ &\leq_{N^n} C\rho_2 + D\rho_2, \text{ by induction hypothesis} \\ &= (C+D)\rho_2 \\ &= B\rho_2. \end{aligned}$$

Case (ii): Let $B = CD$.
Then,

$$\begin{aligned} B\rho_1 &= (CD)\rho_1 \\ &= C(D\rho_1) \\ &\leq_{N^n} C(D\rho_2), \text{ by induction hypothesis} \\ &= (CD)\rho_2 \\ &= B\rho_2. \end{aligned}$$

Therefore, $B\rho_1 \leq_{N^n} B\rho_2$.

(2) Let $A \leq_n B$ in $M_n(N)$ and $\overline{0} \leq_{N^n} \rho$ in $(N^n, +)$. Then by definition of order in $M_n(N)$, we have $\pi_i(A\rho) \leq \pi_i(B\rho)$, for all $\rho \in \mathcal{P}_{N^n}$, $1 \leq i \leq n$. Hence, $A\rho \leq_{N^n} B\rho$. Therefore, N^n is a p.o. $M_n(N)$-group N^n. □

Definition 2.6. Let N^n be a $M_n(N)$-group and let L^n be an ideal of N^n. We say that L^n is convex if $\rho_1, \rho_2 \in L^n$ and $\rho_1 \leq_{N^n} \delta \leq_{N^n} \rho_2$, then $\delta \in L^n$.

Lemma 2.7. If L is a convex ideal in N, then L^n is convex in $M_n(N)$-group N^n.

P r o o f. Suppose that L is a convex ideal in N and let $\rho_1, \rho_1 \in L^n$ and $\delta \in N^n$ such that $\rho_1 \leq_{N^n} \delta \leq_{N^n} \rho_2$. Then, $\pi_i(A\rho_1) \leq \pi_i(A\delta) \leq \pi_i(A\rho_2)$, for all $A \in M_n(N)$, $1 \leq i \leq n$. Since L is convex, we have $\pi_i(A\delta) \in L$, for all $1 \leq i \leq n$. This implies, $A\delta \in L^n$, for all $A \in M_n(N)$. Now for $A = f^1_{ii}$ and $\delta = \langle x_1 \cdots, x_n\rangle$, we have $A\delta = f^1_{ii}\langle x_1, \cdots, x_n\rangle = \langle 0, \cdots, x_i, \cdots, 0\rangle$, implies $\pi_i(A\delta) = x_i \in L$, for all i. Therefore, $\delta = \langle x_1, \cdots, x_n\rangle \in L^n$. □

Definition 2.8. [4] For any ideal $\mathcal{I}$ of N^n,

$$\mathcal{I}_{**} = \{x \in N : x = \pi_j A, \text{ for some } A \in \mathcal{I}, \ 1 \leq j \leq n\},$$

where π_j is the j^{th} projection map from N^n to N.

LEMMA 2.9. [4] If $\mathcal{I}$ is an ideal of $M_n(N)$-group N^n, then $\mathcal{I}_{**} = \{x \in N \mid \langle x, \cdots, 0\rangle \in \mathcal{I}\}$ is a left ideal of N.

LEMMA 2.10. ([4])

(i) If L is an ideal of N^n, then $(L_{**})^n = L$.
(ii) If K is an ideal of ${}_NN$, then $K = (K^n)_{**}$.

LEMMA 2.11. If $\mathcal{L}$ is a convex ideal of $M_n(N)$-group N^n, then $\mathcal{L}_{**}$ is a convex ideal of N.

P r o o f. Let $a \le x \le b$ for $a, b \in \mathcal{L}_{**}$ and $x \in N$. Then $\langle a, 0, \cdots 0\rangle$, $\langle b, 0, \cdots 0\rangle \in \mathcal{L}$. Since $x \in N$, $\langle x, 0, \cdots 0\rangle = f_{11}^{x}\langle 1, \cdots, 0\rangle \in N^n$. Again, since $\langle a, 0, \cdots 0\rangle \le \langle x, 0, \cdots 0\rangle \le \langle b, 0, \cdots 0\rangle$ and $\mathcal{L}$ is convex, we get $\langle x, 0, \cdots 0\rangle \in \mathcal{L}$. Hence, $x \in \mathcal{L}_{**}$. □

THEOREM 2.12. There is a one-one correspondence between the p.o. convex ideals of ${}_NN$ and those of $M_n(N)$-group N^n.

P r o o f. Write $\mathcal{P} = \{I \trianglelefteq_N N : I \text{ is a convex ideal}\}$ and $\mathcal{Q} = \{\mathcal{I} \trianglelefteq_{M_n(N)} N^n : \mathcal{I} \text{ is a convex ideal}\}$. Define

$$\Phi : \mathcal{P} \to \mathcal{Q} \text{ by } \Phi(I) = I^n, \text{ and } \psi : \mathcal{Q} \to \mathcal{P} \text{ by } \psi(\mathcal{I}) = \mathcal{I}_{**}.$$

By Lemma 2.7 and Lemma 2.11, $\Phi(I)$ and $\psi(\mathcal{I})$ are convex ideals of N^n and ${}_NN$ respectively. Now

$$\begin{aligned}(\Phi \circ \psi)(\mathcal{I}) &= \Phi(\mathcal{I}_{**}) \\ &= (\mathcal{I}_{**})^n \\ &= \mathcal{I} \text{ (Lemma 2.10 (i))},\end{aligned}$$

and

$$\begin{aligned}(\psi \circ \Phi)(I) &= \psi(I^n) \\ &= (I^n)_{**} \\ &= I \text{ (Lemma 2.10 (ii))}.\end{aligned}$$

Therefore, $(\Phi \circ \psi) = id_{\mathcal{Q}}$, and $(\psi \circ \Phi) = id_{\mathcal{P}}$, concludes that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic. □

PROPOSITION 2.13. If $f :_N N \to_N \overline{N}$ be an N-isomorphism. Then $\psi : N^n \to (\overline{N})^n$ be an $M_n(N)$-isomorphism defined by $\psi(\rho) = \overline{\rho}$, that is, if $\rho = \langle a_1, \cdots, a_n\rangle \in N^n$, then $\overline{\rho} = \langle f(a_1), \cdots, f(a_n)\rangle = \langle \overline{a_1}, \cdots, \overline{a_n}\rangle$, where N^n and $\overline{N}^n$ are $M_n(N)$-group N^n and $M_n(\overline{N})$-group $\overline{N}^n$, respectively.

P r o o f. (i) Let $\rho_1, \rho_2 \in N^n$, where $\rho_1 = \langle a_1, \cdots, a_n\rangle$ and $\rho_2 = \langle b_1, \cdots, b_n\rangle$.

Then,

$$\begin{aligned}
\psi(\rho_1 + \rho_2) &= \psi(\langle a_1, \cdots, a_n\rangle + \langle b_1, \cdots, b_n\rangle) \\
&= \psi(\langle a_1 + b_1, \cdots, a_n + b_n\rangle) \\
&= \langle f(a_1 + b_1), \cdots, f(a_n + b_n)\rangle \\
&= \langle f(a_1) + f(b_1), \cdots, f(a_n) + f(b_n)\rangle \\
&= \langle \overline{a_1} + \overline{b_1}, \cdots, \overline{a_n} + \overline{b_n}\rangle \\
&= \langle \overline{a_1}, \cdots, \overline{a_n}\rangle + \langle \overline{b_1}, \cdots, \overline{b_n}\rangle \\
&= \langle f(a_1), \cdots, f(a_n)\rangle + \langle f(b_1), \cdots, f(b_n)\rangle \\
&= \psi(\langle a_1, \cdots, a_n\rangle) + \psi(\langle b_1, \cdots, b_n\rangle) \\
&= \psi(\rho_1) + \psi(\rho_2)
\end{aligned}$$

(ii) Let $A \in M_n(N)$, $\rho \in N^n$. Let $A = f_{ij}^x$, where $x \in N$, and $\rho = \langle a_1, \cdots, a_n\rangle$.

Then,

$$\begin{aligned}
\psi(A\rho) &= \psi(f_{ij}^x\langle a_1, \cdots, a_n\rangle) \\
&= \psi(\langle 0, \cdots, \underbrace{xa_j}_{i^{th}}, \cdots, 0\rangle) \\
&= \langle f(0), \cdots, \underbrace{f(xa_j)}_{i^{th}}, \cdots, f(0)\rangle \\
&= \langle \overline{0}, \cdots, \underbrace{xf(a_j)}_{i^{th}}, \cdots, \overline{0}\rangle \text{ (since } f \text{ is a homomorphism)} \\
&= \langle \overline{0}, \cdots, \underbrace{x\overline{a_j}}_{i^{th}}, \cdots, \overline{0}\rangle
\end{aligned}$$

$$\begin{aligned}
&= f_{ij}^{x}(\langle \overline{a_1}, \cdots, \overline{a_j}, \cdots, \overline{a_n} \rangle) \\
&= f_{ij}^{x}\langle f(a_1), \cdots, f(a_n) \rangle \\
&= f_{ij}^{x}\psi(\langle a_1, \cdots, a_n \rangle) \\
&= f_{ij}^{x}\psi(\rho) \\
&= A\psi(\rho).
\end{aligned}$$

Assume that the result is true for $w(A) < k$. Suppose that $w(A) = k$. Then $A = C + D$ or $A = CD$, where $w(C) < k,\ w(D) < k$.

Case-(i): $A = C + D$. Then, $\psi(A\rho) = \psi((C + D)\rho) = \psi(C\rho + D\rho)$. Let $C\rho = \langle c_1, \cdots, c_n \rangle$ and $D\rho = \langle d_1, \cdots, d_n \rangle$.

Now,

$$\begin{aligned}
\psi(C\rho + D\rho) &= \psi(\langle c_1, \cdots, c_n \rangle + \langle d_1, \cdots, d_n \rangle) \\
&= \psi(\langle c_1 + d_1, \cdots, c_n + d_n \rangle) \\
&= \langle f(c_1 + d_1), \cdots, f(c_n + d_n) \rangle \\
&= \langle f(c_1) + f(d_1), \cdots, f(c_n) + f(d_n) \rangle \\
&= \langle \overline{c_1} + \overline{d_1}, \cdots, \overline{c_n} + \overline{d_n} \rangle \\
&= \langle \overline{c_1}, \cdots, \overline{c_n} \rangle + \langle \overline{d_1}, \cdots, \overline{d_n} \rangle \\
&= \langle f(c_1), \cdots, f(c_n) \rangle + \langle f(d_1), \cdots, f(d_n) \rangle \\
&= \psi(\langle c_1, \cdots, c_n \rangle) + \psi(\langle d_1, \cdots, d_n \rangle) \\
&= \psi(C\rho) + \psi(D\rho),\ \text{as } w(C), w(D) < k \\
&= (C + D)\psi(\rho) \\
&= A\psi(\rho)
\end{aligned}$$

Case-(ii): $A = CD$.

Then,

$$\begin{aligned}
\psi(A\rho) &= \psi((CD)\rho) \\
&= \psi(C(D\rho)) \\
&= C\psi(D\rho),\ \text{as } w(C) < k,\ D\rho \in N^n \\
&= CD\psi(\rho),\ \text{as } w(D) < k,\ \rho \in N^n. \\
&= A\psi(\rho)
\end{aligned}$$

Therefore, ψ is a homomorphism.

Now, to show ψ is one-one, let $\rho_1, \rho_2 \in N^n$, where $\rho_1 = \langle a_1, \cdots, a_n \rangle$, $\rho_2 = \langle b_1, \cdots, b_n \rangle$ such that $\psi(\rho_1) = \psi(\rho_2)$.

Then, $\psi(\langle a_1, \cdots, a_n\rangle) = \psi(\langle b_1, \cdots, b_n\rangle)$ implies $\langle f(a_1), \cdots, f(a_n)\rangle = \langle f(b_1), \cdots, f(b_n)\rangle$. □

3. Conclusion

We have defined the partial order in $M_n(N)$-group N^n, and proved one-one correspondence between the convex ideal of N-group (over itself) and the $M_n(N)$-group N^n. We also have characterized positive cone in module over a matrix nearring $M_n(N)$-group N^n. This can be extended to study lattice order in matrix nearrings and related properties.

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