

A NOTE ON THE FAITHFULNESS OF
THE EVALUATED GASSNER REPRESENTATION
OF THE PURE BRAID GROUP

Mohamad N. Nasser

¹ Department of Mathematics and Computer Science

Beirut Arab University

P.O. Box 11-5020, Beirut, LEBANON

e-mail: m.nasser@bau.edu.lb

Abstract

Bureau representation of the braid group, B_n , has been proved to be faithful for $n \leq 3$ and unfaithful for $n \geq 5$; whereas the case $n = 4$ remains open. On the other hand, the question of faithfulness of Gassner representation of the pure braid group, P_n , is still open for $n \geq 4$. For any $n \geq 4$, T. Chuna specified a family of new non-trivial elements in the kernel of the evaluated Bureau representation at m^{th} root of unity, where $m \in \mathbb{N}^* - \{3\}$. Along the same lines as Chuna's work, we find, for $n \geq 4$, a family of new non-trivial elements in the kernel of the evaluated Gassner representation at m^{th} root of unity, where $m \in \mathbb{N}^* - \{3\}$. This result may possibly be a road toward answering the open question of the faithfulness of Gassner representation.

Math. Subject Classification: Primary 20F36

Key Words and Phrases: braid group, pure braid group, Bureau representation, Gassner representation, faithfulness

1. Introduction

The braid group on n strings, B_n , is the abstract group, introduced by E. Artin in [3], which is generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ with the following presentation:

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i &= 1, 2, \dots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i-j| &\geq 2.\end{aligned}$$

The pure braid group, P_n , is a normal subgroup of B_n , which is defined as the kernel of the homomorphism $B_n \rightarrow S_n$ defined by $\sigma_i \rightarrow (i \ i+1)$, $1 \leq i \leq n-1$, where S_n is the symmetric group of n elements. It admits a presentation with the following generators.

$$A_{ij} = \sigma_{j-1} \sigma_{j-2} \dots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}, \quad 1 \leq i < j \leq n.$$

The most famous linear representations of B_n are Burau representation [7] and Lawrence-Krammer-Bigelow representation [12]. Burau representation has been proved to be faithful for $n \leq 3$ [6] and unfaithful for $n \geq 5$ in [14, 13, 5]; whereas the case $n = 4$ remains open. Lawrence-Krammer-Bigelow representation has been proved to be faithful for all n [4]; which shows that B_n is linear.

On the other hand, the most famous linear representation of P_n is Gassner representation [6]. Gassner representation is faithful for $n \leq 3$ due to the faithfulness of Burau representation for $n \leq 3$. However, the question of whether or not Gassner representation of P_n is faithful still an open question for $n \geq 4$. M. Abdulrahim proved that an element of P_n lies in the kernel of the reduced Gassner representation if and only if the trace of its image is equal to that of the identity matrix [1].

In [8], T. Chuna proved, for $n \geq 4$, that the evaluated Burau representation of B_n at m^{th} root of unity with $m \in \mathbb{N}^* - \{3\}$ is unfaithful by considering a family of new non-trivial elements in its kernel. More precisely, T. Chuna proved that, for all $n \geq 4$ and $t = e^{\frac{2i\pi}{m}}$ with $m \in \mathbb{N}^* - \{3\}$, there exists an even integer k such that the elements $(\sigma_i \sigma_{i+1} \sigma_i)^k$ are in the kernel of the evaluated Burau representation. We cannot know whether this family of elements are in the kernel of the evaluated Gassner representation of P_n , for $n \geq 4$, or not. For that, the aim of this paper is to prove that, for all $n \geq 4$, the evaluated Gassner representation of P_n at $e^{\frac{2i\pi}{m}}$ with $m \in \mathbb{N}^* - \{3\}$ is unfaithful by finding a family of new non-trivial elements that belong to its kernel. Finding the shape of such

family of elements may possibly be a road toward answering the question of whether or not the Gassner representation of P_n is faithful.

In more depth, the main theorem in this paper is Theorem 3.1, where we prove that, for all $n \geq 4$ and for $t_1 = t_2 = \dots = t_n = e^{\frac{2i\pi}{m}}$ with $m \in \mathbb{N}^* - \{3\}$, there exists an integer s such that the elements $(A_{i,i+2}A_{i+1,i+2}A_{i,i+1})^s$ are non-trivial elements in the kernel of the evaluated Gassner representation of P_n for all $1 \leq i \leq n-2$.

2. Preliminaries

The question of faithfulness of representations of the braid group B_n and its normal subgroup P_n has been always of a lot of significance to people working on representation theory. Burau representation of B_n has been proved to be faithful for $n \leq 3$ and unfaithful for $n \geq 5$; while the case $n = 4$ remains open. The faithfulness of Gassner representation of P_n still unknown for $n \geq 4$. Regarding irreducibility of these representations, previous work has been done to determine conditions under which Burau and Gassner representations are irreducible. The criteria for the irreducibility of Burau representation have been found by E. Formanek [10]; while that for Gassner representation was found by M. Abdulrahim [2].

In what follows, we recall main definitions of Burau and Gassner representations, and we introduce some basic results on them.

DEFINITION 2.1. [7] Let t be indeterminate. Burau representation $\beta_n(t) : B_n \rightarrow GL_n(\mathbb{Z}[t^{\pm 1}])$ is defined by

$$\sigma_i \rightarrow \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 1-t & t & 0 \\ & 1 & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right) \text{ for } i = 1, 2, \dots, n-1.$$

DEFINITION 2.2. Burau representation $\beta_n(t)$ is called "evaluated Burau representation" if t is evaluated at some value.

THEOREM 2.1. [10] *Burau representation is reducible. Moreover, the reduced Burau representation $\beta_n(z) : B_n \rightarrow GL_{n-1}(\mathbb{C})$, $z \in \mathbb{C}$, is irreducible if and only if z is not a root of the polynomial $f_n(x) = x^{n-1} + x^{n-2} + \dots + x + 1$.*

DEFINITION 2.3. Let $t = (t_1, t_2, \dots, t_n)$ where each t_i is indeterminate. Gassner representation $\gamma_n(t) : P_n \rightarrow GL_n(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}])$ is defined by

$$A_{i,j} \rightarrow I_n - \begin{pmatrix} 0 & 0 & 0 \\ 0 & S_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$S_{ij} = \begin{pmatrix} 1 - t_j & 0 & \dots & 0 & -1 + t_j \\ (1 - t_i)(1 - t_j) & 0 & \dots & 0 & (-1 + t_i)(1 - t_j) \\ \vdots & & & & \vdots \\ (1 - t_i)(1 - t_j) & 0 & \dots & 0 & (-1 + t_i)(1 - t_j) \\ (1 - t_i)(-t_j) & 0 & \dots & 0 & (-1 + t_i)(-t_j) \end{pmatrix}.$$

DEFINITION 2.4. Gassner representation $\gamma_n(t)$ where $t = (t_1, t_2, \dots, t_n)$ is called "evaluated Gassner representation" if each t_i is evaluated at some value.

THEOREM 2.2. [2] *Gassner representation is reducible. Moreover, the reduced Gassner representation $\gamma_n(z) : P_n \rightarrow GL_{n-1}(\mathbb{C})$, $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, is irreducible if and only if $z_1 z_2 \dots z_n \neq 1$.*

LEMMA 2.1. [15] *If the t'_i s in Gassner representation are all specialized to the same indeterminate t , then Gassner representation becomes the restriction of Burau representation of B_n on P_n .*

Now, we give some results on the faithfulness of Gassner representation.

THEOREM 2.3. [1] *If the trace of the image of an element of P_n under the reduced Gassner representation is $n - 1$, then this element lies in the kernel of this representation.*

PROPOSITION 2.1. *Consider $n \geq 2$ and let $t = (t_1, t_2, \dots, t_n)$ where each t_i is indeterminate. Then, for all $1 \leq i < j \leq n$, $A_{i,j}^k \notin \ker(\gamma_n(t))$ for all $k \in \mathbb{Z}^*$.*

P r o o f. The proof is straightforward from the fact that, for all $1 \leq i < j \leq n$, $\det(A_{i,j}^k) = t_i^k t_j^k \neq 1$ for all $k \in \mathbb{Z}^*$. $\square$

THEOREM 2.4. Consider $n \geq 2$ and let $t = (t_1, t_2, \dots, t_n)$ where each t_i is indeterminate. Let w be a non-trivial element in P_n consists of the product of the elements $A_{i_1, j_1}, A_{i_2, j_2}, \dots, A_{i_l, j_l}$, where $i_1 < j_1 < i_2 < j_2 < \dots < i_l < j_l$. Then, $w \notin \ker(\gamma_n(t))$.

P r o o f. As the elements $A_{i_1, j_1}, A_{i_2, j_2}, \dots, A_{i_l, j_l}$, $i_1 < j_1 < i_2 < j_2 < \dots < i_l < j_l$, commute, we can write $w = A_{i_1, j_1}^{k_1} A_{i_2, j_2}^{k_2} \dots A_{i_l, j_l}^{k_l}$, where k_s is the number of times A_{i_s, j_s} occurs in w . Clearly not all k_s are zero as w is non-trivial. Thus, $\det(w) = t_{i_1}^{k_1} t_{j_1}^{k_1} t_{i_2}^{k_2} t_{j_2}^{k_2} \dots t_{i_l}^{k_l} t_{j_l}^{k_l} \neq 1$, and therefore $w \notin \ker(\gamma_n(t))$, as required. $\square$

3. The evaluated Gassner representation at m^{th} root of unity

In order to determine the faithfulness of representations of braid groups, Garside found in [11] the shape of the center of the braid group B_n for $n \in \mathbb{N}^*$. He showed that, for all $n \in \mathbb{N}^*$, B_n has an infinite cyclic center $Z(B_n)$ generated by the element $c = (\sigma_1 \sigma_2 \dots \sigma_{n-1})^n$. On the other hand, B. Farb and D. Margalit [9] showed that c is also an element in the center of the pure braid group P_n , $n \in \mathbb{N}^*$, which means that $Z(B_n) = Z(P_n)$. More precisely, for each $n \in \mathbb{N}^*$, P_n has an infinite cyclic center $Z(P_n)$ generated by the element $c = (\sigma_1 \sigma_2 \dots \sigma_{n-1})^n = (A_{1,2} A_{1,3} \dots A_{1,n})(A_{2,3} A_{2,4} \dots A_{2,n}) \dots (A_{n-1,n})$.

In what follows, we prove that, for any $n \in \mathbb{N}^*$, the evaluated Burau and Gassner representations at m^{th} root of unity are unfaithful for any non-zero integer m .

PROPOSITION 3.1. Specialize $t, t_1, t_2, \dots, t_n$ to be non-zero complex numbers $z, z_1, z_2, \dots, z_n$ respectively. Then, the following holds true.

- (a) If $z = e^{\frac{2i\pi}{m}}$, with $m \in \mathbb{Z}^*$, then the evaluated Burau representation $\beta_n(z)$ is unfaithful.
- (b) If $z_1 = z_2 = \dots = z_n = e^{\frac{2i\pi}{m}}$, with $m \in \mathbb{Z}^*$, then the evaluated Gassner representation $\gamma_n(z_1, z_2, \dots, z_n)$ is unfaithful.

P r o o f. For the proof of (a), we have $c = (\sigma_1 \sigma_2 \dots \sigma_{n-1})^n$ with $\beta_n(c^m) = z^{nm} I_n = (z^m)^n I_n = I_n$ for any $m \in \mathbb{Z}^*$, since z is the m^{th} root of unity. Hence, c is a non-trivial element in the kernel of $\beta_n(z)$, and

so $\beta_n(z)$ is unfaithful. The proof of (b) is a consequence from (a), as $Z(B_n) = Z(P_n)$, and from Lemma 2.1. $\square$

Now, T. Chuna proved that, for all $n \geq 4$ and $t = e^{\frac{2i\pi}{m}}$ with $m \in \mathbb{N}^* - \{3\}$, there exists an even integer k such that the elements $(\sigma_i \sigma_{i+1} \sigma_i)^k$ are non-trivial elements in the kernel of the evaluated Burau representation of B_n [8]. We cannot know whether this element is in the kernel of the evaluated Gassner representation of P_n or not. In the next theorem, we find a family of new non-trivial elements that lie in the kernel of the evaluated Gassner representation of P_n for all $n \geq 4$. So, we prove a similar result to Chuna's result, but for the evaluated Gassner representation.

THEOREM 3.1. *Specialize $t_1, t_2, \dots, t_n$ to be non-zero complex numbers $z_1, z_2, \dots, z_n$ respectively and set $z_1 = z_2 = \dots = z_n = e^{\frac{2i\pi}{m}}$ with $m \in \mathbb{N}^* - \{3\}$. Then, for all $n \geq 4$, there exists an integer s such that the elements $(A_{i,i+2}A_{i+1,i+2}A_{i,i+1})^s$ are non-trivial elements in the kernel of the evaluated Gassner representation of P_n for all $1 \leq i \leq n-2$.*

P r o o f. For a fixed $1 \leq i \leq n-2$, consider the element $w_i = A_{i,i+2}A_{i+1,i+2}A_{i,i+1}$. First of all, we use induction to prove that, for any $h \in \mathbb{N}^*$, we have

$$w_i^h = \sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)^{2h-1}\sigma_i. \quad (1)$$

Indeed, for $h = 1$, we have

$$\begin{aligned} w_i^1 &= A_{i,i+2}A_{i+1,i+2}A_{i,i+1} \\ &= (\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1})(\sigma_{i+1}^2)(\sigma_i^2) \\ &= \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\sigma_{i+1}^2\sigma_i^2 \\ &= \sigma_{i+1}\sigma_i^2\sigma_{i+1}\sigma_i^2 \\ &= \sigma_{i+1}\sigma_i\sigma_i\sigma_{i+1}\sigma_i\sigma_i \\ &= \sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)\sigma_i, \text{ as claimed in (1).} \end{aligned}$$

Now, suppose that (1) is true for h and let us prove that it is true also for $h+1$. We have

$$\begin{aligned} w_i^{h+1} &= w_i^h.w_i = [\sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)^{2h-1}\sigma_i][\sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)\sigma_i] \\ &= \sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)^{2h-1}\sigma_i\sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)\sigma_i \\ &= \sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)^{2h-1}(\sigma_i\sigma_{i+1}\sigma_i)(\sigma_i\sigma_{i+1}\sigma_i)\sigma_i \\ &= \sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i)^{2h+1}\sigma_i, \text{ as claimed in (1).} \end{aligned}$$

Thus, the equality (1) holds for any $h \in \mathbb{N}^*$.

Recall that, by T. Chuna in [8], there exists an even integer k such that the element $(\sigma_i \sigma_{i+1} \sigma_i)^k$ is in the kernel of the evaluated Burau representation; that is $\beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^k] = I_n$. Set $k = 2s$ as k is even. Require to prove that $\gamma_n(w_i^s) = I_n$. Indeed, we have

$$\begin{aligned}
 \gamma_n(w_i^s) &= \gamma_n[\sigma_{i+1} \sigma_i (\sigma_i \sigma_{i+1} \sigma_i)^{2s-1} \sigma_i] \\
 &= \beta_n[\sigma_{i+1} \sigma_i (\sigma_i \sigma_{i+1} \sigma_i)^{2s-1} \sigma_i], \text{ (using Lemma 2.1)} \\
 &= \beta_n(\sigma_{i+1} \sigma_i) \beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^{2s-1}] \beta_n(\sigma_i) \\
 &= \beta_n(\sigma_{i+1} \sigma_i) \beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^{2s}] \beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^{-1}] \beta_n(\sigma_i) \\
 &= \beta_n(\sigma_{i+1} \sigma_i) \beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^k] \beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^{-1}] \beta_n(\sigma_i) \\
 &= \beta_n(\sigma_{i+1} \sigma_i) I_n \beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^{-1}] \beta_n(\sigma_i) \\
 &= \beta_n(\sigma_{i+1} \sigma_i) \beta_n[(\sigma_i \sigma_{i+1} \sigma_i)^{-1}] \beta_n(\sigma_i) \\
 &= \beta_n(\sigma_{i+1} \sigma_i) \beta_n(\sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1}) \beta_n(\sigma_i) \\
 &= \beta_n(\sigma_{i+1} \sigma_i \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_i) \\
 &= \beta_n(id_{B_n}) = I_n, \text{ where } id_{B_n} \text{ is the identity element of } B_n.
 \end{aligned}$$

Now, suppose to get a contradiction that, for $1 \leq i \leq n-2$, w_i^s is a trivial element in P_n ; that is $w_i^s = id_{P_n}$, where id_{P_n} is the identity element of P_n . Then, we get that $\sigma_{i+1} \sigma_i (\sigma_i \sigma_{i+1} \sigma_i)^{2s-1} \sigma_i$ is a trivial element in B_n ; that is $\sigma_{i+1} \sigma_i (\sigma_i \sigma_{i+1} \sigma_i)^{2s-1} \sigma_i = id_{B_n}$, and so, $(\sigma_i \sigma_{i+1} \sigma_i)^{2s-1} = \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} = (\sigma_i \sigma_{i+1} \sigma_i)^{-1}$. Hence, $(\sigma_i \sigma_{i+1} \sigma_i)^{2s} = id_{B_n}$, and so $(\sigma_i \sigma_{i+1} \sigma_i)^k = id_{B_n}$ which contradicts Chuna's result. Thus w_i^s is a non-trivial element in P_n for all $1 \leq i \leq n-2$.

Therefore, for a certain integer $s = \frac{k}{2}$, the elements $(A_{i,i+2} A_{i+1,i+2} A_{i,i+1})^s$ are non-trivial elements in the kernel of the evaluated Gassner representation for all $1 \leq i \leq n-2$ and the proof is completed. $\square$

References

- [1] M. Abdulrahim, A faithfulness criterion for the Gassner representation of the pure braid group, *Proc. of the Amer. Math. Soc.*, **125** (1997), 1249-1257.
- [2] M. Abdulrahim, Complex specializations of the reduced Gassner representation of the pure braid group, *Proc. of the Amer. Math. Soc.*, **125** (1997), 1617-1624.
- [3] E. Artin, Theorie der Zöpfe, *Abhandlungen Hamburg*, **4** (1925), 47-72.
- [4] S. Bigelow, Braid groups are linear, *J. Amer. Math. Soc.*, **14** (2001), 471-486.

- [5] S. Bigelow, The Burau representation is not faithful for $n = 5$, *Topology*, **32** (1999), 439-447.
- [6] J.S. Birman, *Braids, Links and Mapping Class Groups*, Annals of Mathematical Studies, Princeton University Press (1974).
- [7] W. Burau, Braids, Uber Zopfgruppen und gleichsinnig verdrillte Verkettungen, *Abh. Math. Semin. Hamburg Univ.*, **11** (1936), 179-186.
- [8] T. Chuna, *The Evaluated Burau Representation of the Braid Group*, Thesis in Wittenberg University (2016); <https://api.semanticscholar.org/CorpusID:46915134>.
- [9] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press (2012).
- [10] E. Formanek, Braid group representations of low degree, *Proc. London Math. Soc.*, **3**, No 2 (1996), 279-322.
- [11] F. Garside, The Braid group and other groups, *The Quarterly Journal of Mathematics*, **20** (1969), 235-254.
- [12] D. Krammer, Braid groups are linear, *Annals Math.*, **155** (2002), 131-156.
- [13] D. Long, M. Paton, The Burau representation of the braid group B_n is not faithful for $n \geq 6$, *Topology*, **32** (1992), 439-447.
- [14] J. Moody, The Burau representation of the braid group B_n is not faithful for large n , *Bull. Amer. Math. Soc.*, **25** (1991), 379-384.
- [15] R. Smeltzer, *Linear Representation of the Braid Group*, Thesis in McMaster University (2016); <https://math.mcmaster.ca/~boden/students/Smeltzer-MSc.pdf>.