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ON ONE HIGHLY ACCURATE AND EFFICIENT METHOD FOR SOLVING THE BIHARMONIC EQUATION

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Abstract

In this paper, a discrete version of the preliminary integration method for the numerical solution of the boundary value problem of a non-homogeneous biharmonic equation is proposed. Partial derivatives of the equation are presented in the form of finite double series in Chebyshev polynomials of the first kind with unknown expansion coefficients. Using discrete integration formulas that reduce the order of derivatives, the main equation is "*integrated*" four times both with respect to the variable "x" and the variable "y". By adding boundary conditions written in the form of finite double series to the resulting equation, an algebraic system is formed for determining the unknown coefficients. The numerical calculations performed with the selected various trial functions show the high accuracy and efficiency of the proposed method.

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1. **Introduction**

Biharmonic equations are often encountered in many areas of engineering and physics, describing some phenomena and have numerous applications. For this purpose, we will provide a brief overview of literature sources illustrating the importance of biharmonic equations and methods for solving these equations.

In the article [2] it is stated that the biharmonic function plays a major role in elasticity theory. For example, when solving a plane problem of elasticity theory using a known biharmonic function, one can immediately obtain the stress state using the Airy formulas. The author of the article [8] considers the application of approximation theory methods to optimality principles in decision theory; in this regard, the asymptotic properties of solutions to biharmonic equations as approximate functions are investigated. In the article [9], a complex inhomogeneous biharmonic equation is used to describe a hydroelastic problem of free oscillations of a thin plate horizontally separating ideal incompressible fluids of different densities in a rigid cylindrical reservoir of arbitrary cross-section. The article [11] considers an optimal control problem described by a biharmonic equation with limited boundary conditions. To address the problem of parameter sensitivity and unsatisfactory accuracy for physics arising in the areas of scientific computing and engineering applications, the authors of [12] propose a method for approximating the solution for a class of fourth-order biharmonic equations with two types of boundary conditions in unified and non-unified domains. As an application, the solution of the inhomogeneous biharmonic equation is used to model the stress-strain state of an isotropic elastic thin plate of polygonal shape under the action of a transverse load [19]. Advection-diffusion phenomena governing the transfer of chemical substances in porous media, when diffusion processes are considered, the transfer process is described by the biharmonic equation [21].

The article [3] discusses solution of the biharmonic equation in mixed form discretized by a high-order hybrid method. Numerical experiments evaluating the efficiency of the proposed iterative algorithm are presented. Despite the fact that biharmonic equations have many applications in solid and fluid mechanics, they are difficult to solve due to the presence of fourth-order partial derivatives [4]. A constructive presentation of the pre-integration method for solving a homogeneous biharmonic equation is given in [7]. In [17], a compact fourth-order finite-difference scheme is proposed for solving biharmonic equations with Dirichlet boundary conditions. In [18], a step-by-step construction of a finite-difference scheme for an inhomogeneous biharmonic equation with homogeneous boundary conditions imposed on the desired function and its first-order partial derivatives is presented. The finite-difference scheme is based on a square twenty-five-point template and has an implicit character and approximates the boundary value problem with the second order of accuracy in the grid step. The author claims that the obtained results correspond to the physical meaning of the problem and are consistent with similar numerical and approximate-analytical solutions. In the article [20], a new version of the collocation method and least squares of increased accuracy for the numerical solution of the inhomogeneous biharmonic equation is proposed and implemented. In numerical experiments, it was found that the solution converges with an increased order and coincides with the analytical solution of the problem with high accuracy. In the article [23], the corresponding discontinuous Galerkin finite element method is used to solve the biharmonic equation. This method, judging by its name, uses discontinuous approximations and at the same time preserves the simple formulation of the corresponding finite element method. The use of the C-dependent discontinuous Galerkin finite element method for solving the biharmonic equation is described in [24]. The Galerkin finite element method without a stabilizer was presented and analyzed in [25] for the biharmonic equation, which has an extremely simple formulation from finite elements.

When solving harmonic equations (Laplace, Poisson) by numerical methods, a system of linear algebraic equations is obtained, the matrix of which has a very large order. In addition, the matrix of the algebraic system is sparse, i.e. it has many zero elements and, finally, it is an ill-conditioned matrix, i.e. the ratio of the largest eigenvalue of the matrix to its smallest eigenvalue is very large. In the case of biharmonic (doubly harmonic) equations, the situation is aggravated, the numerical solution of such equations encounters serious difficulties and many numerical methods become practically uneconomical. Iterative methods can be used to solve biharmonic equations, however, the number of iterations in which is often called very large.

In this regard, the development of a highly accurate and efficient direct method for the numerical solution of the inhomogeneous biharmonic equation is of undoubted interest.

Along with the above methods, it is possible to apply a discrete version of the preliminary integration method [1, 13, 16]. The main advantage of the proposed method is the high rate of convergence of the calculation results. Due to the high accuracy of the method, when using it with a smaller number of Chebyshev polynomials, it is possible to achieve the same accuracy as in other methods. Or, in other words, with the same amount of calculations, it is possible to study a biharmonic equation with higher values of characteristic parameters. To ensure the efficiency of the method when calculating double finite series by Chebyshev polynomials, it is possible to apply the fast discrete Fourier cosine transform [5, 6, 10]. The use of Chebyshev polynomials for numerical modeling of the eigenvalue problem for a nonlinear ordinary differential equation with a small parameter at the highest derivative and for a system of similar equations is given in [14, 15]. The idea of choosing trial functions is presented in [12].

2. **Statement of the problem**

A numerical solution of the biharmonic equation be required in the region $\overline{D} = \{-1 \leq x, y \leq 1\}$

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = -f(x, y), \quad -1 < x, y < 1, \tag{1}$$

with the following boundary conditions:

$$\begin{aligned} & u(-1, y) = 0, u(1, y) = 0, \\ & u(x, -1) = 0, u(x, 1) = 0, \\ & \tfrac{\partial u}{\partial x}(-1, y) = 0, \tfrac{\partial u}{\partial x}(1, y) = 0, \\ & \tfrac{\partial u}{\partial y}(x, -1) = 0, \tfrac{\partial u}{\partial y}(x, 1) = 0, \end{aligned} \tag{2}$$

where $f(x, y)$ is a given function determined by the following formula

$$f(x, y) = -(\frac{\partial^4 u_e}{\partial x^4} + 2\frac{\partial^4 u_e}{\partial x^2 \partial y^2} + \frac{\partial^4 u_e}{\partial y^4}). \tag{3}$$

For the numerical solution of problem (1) - (2), a discrete version of the preliminary integration method is used. To check the convergence and order of accuracy of the method used, we use the method of trial functions [22]. The essence of this method is as follows. A certain function is selected $u_e(x,y) = u_{exact}(x,y)$, it can be chosen arbitrarily, but so that the boundary conditions (2) are satisfied. Substituting it into equation (1), we find the right-hand side of (3). The resulting problem is solved by a discrete version of the preliminary integration method, and the approximate solution $u_a(x,y) = u_{approximate}(x,y)$ is compared with the known function $u_e(x,y)$ in the collocation nodes of the Chebyshev polynomials (x_l, y_k), where $x_l = \cos\frac{\pi l}{N}, (l = 0, 1, ..., N)$, $y_k = \cos\frac{\pi k}{M}, (k = 0, 1, ..., M)$. Thus, as trial functions for problem (1) - (2), we consider two functions of the following type:

$$\begin{aligned} u_e^{(1)}(x,y) &= (1-x^2)^2(1-y^2)^2 e^{A\sin x\sin y}, \\ u_e^{(2)}(x,y) &= (1-x^2)^2(1-y^2)^2 e^{A(x+y)}. \end{aligned} \tag{4}$$

For the selected functions (4), the right-hand sides of the form (3) respectively have the form: for the first trial function $u_e^{(1)}(x,y)$

$$\begin{aligned} & f^{(1)}(x,y) = -e^{A\sin x\sin y}((y^2-1)^2 \times (A\sin y(3(x^2-1)^2 \\ & \times\sin^2 x + \sin x(-6A^2(x^2-1)^2\sin^2 y\cos^2 x \\ & -48Ax(x^2-1)\sin y\cos x + (x^4-74x^2+25)) \\ & +A\sin y\cos^2 x(A^2(x^2-1)^2\sin^2 y\cos^2 x \\ & +16Ax(x^2-1)\sin y\cos x - 4(x^4-20x^2+7)) \\ & +16x(7-x^2)\cos x) + 24) + 2\times((A(y^2-1)^2 \\ & \times(x^2-1)(-2A(x^2-1)\cos 2y\cos^2 x \\ & -\sin x\sin y(x^2-1) + 8x\cos x\sin y)+ \\ & +A(y^2-1)^2\sin x\sin y(A(x^2-1)^2\sin y \\ & \times(-\cos^2 x\sin y + \sin x) + 8Ax(1-x^2)\cos x\sin y \\ & -12x^2+4) - A(y^2-1)\sin x\cos y(A\sin x(y^2-1) \\ & \times\cos y + 8y)(-A^2(x^2-1)^2\cos^2 x\sin^2 y \\ & +A(x^2-1)^2\sin x\sin y - 8Ax(x^2-1)\cos x\sin y \\ & -12x^2+4) - 4(3y^2-1)(-A^2(x^2-1)^2\cos^2 x \\ & \times\sin^2 y + A(x^2-1)^2\sin x\sin y - 8Ax(x^2-1) \\ & \times\cos x\sin y - 12x^2+4) - A(y^2-1)(x^2-1)\cos y \end{aligned}$$

$$
\begin{aligned}
&\times(2A(y^2-1)\sin x\cos y+8y)(-2A(x^2-1)\\
&\times\cos^2 x\sin y+(x^2-1)\sin x-8x\cos x)))\\
&+(x^2-1)^2(A\sin x(3(y^2-1)^2\sin^2 y\\
&+\sin y(-6A^2(y^2-1)^2\sin^2 x\cos^2 y-48Ay\\
&\times(y^2-1)\sin x\cos y+(y^4-74y^2+25))\\
&+A\sin x\cos^2 y(A^2(y^2-1)^2\sin^2 x\cos^4 y\\
&+16Ay(y^2-1)\sin x\cos^3 y-4(y^4-20y^2+7))\\
&+16y(7-y^2)\cos y)+24)),
\end{aligned} \tag{5}
$$

for the second trial function $u_e^{(2)}(x,y)$

$$
\begin{aligned}
&f^{(2)}(x,y)=-(e^{A(x+y)}(A(y^2-1)^2(A^2x^3(Ax+12)\\
&+2A(18-A^2)x^2+12(2-A^2)x+A(A^2-12))\\
&+4(y^2-1)^2(A^2x^2(Ax+9)+A(18-A^2)x\\
&+3(2-A^2)))++2\times(e^{A(x+y)}\times((A^2(x^2-1)\\
&\times(Ax^2+4x-A)+2Ax(Ax^2+4x-A)\\
&+A(2Ax+4)(x^2-1))(y^2-1)^2+4(y^2-1)\\
&\times(4(x^2-1)(Ax^2+4x-A)y+2x(Ax^2+4x-A)y\\
&+(2Ax+4)(x^2-1)y)))+e^{A(x+y)}(A(x^2-1)^2(A^2y^3\\
&\times(Ay+12)+2A(18-A^2)y^2+12(2-A^2)y\\
&+A(A^2-12))+4(x^2-1)^2(A^2y^2(Ay+9)\\
&+A(18-A^2)y+3(2-A^2)))).
\end{aligned}
$$

3. **Solution method**

The essence of the discrete method of preliminary integration is as follows. All partial derivatives in equation (1) and the right-hand side are presented in the form of finite double series in Chebyshev polynomials of the first kind with unknown expansion coefficients. Substituting these series into equation (1) and equating the coefficients at the same degrees of the polynomials, we obtain a system of linear algebraic equations. This system of preliminary fourfold “integrates” with respect to the variable "x" and with respect to the variable "y" using discrete formulas for reducing the degrees of derivatives and a new system of equations is obtained. Adding to this system the equations obtained from the boundary conditions (2) written through finite series in Chebyshev polynomials, we obtain a system of algebraic equations for determining the coefficients of a_{ij} the approximate solution $u_a(x,y)$.

Thus, we have the following finite series:

$$
\begin{aligned}
\frac{\partial^4 u_a}{\partial x^4} &= \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' a_{ij}^{(4x)} T_i(x) T_j(y), \\
\frac{\partial^4 u_a}{\partial x^2 \partial y^2} &= \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' a_{ij}^{(2x,2y)} T_i(x) T_j(y), \\
\frac{\partial^4 u_a}{\partial y^4} &= \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' a_{ij}^{(4y)} T_i(x) T_j(y), \\
f(x,y) &= \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' g_{ij} T_i(x) T_j(y),
\end{aligned} \tag{6}
$$

where the primes above the sums mean that the coefficient a_{ij} is taken with the multiplier $\frac{1}{2}$ when the index i or j is zero, is taken with the multiplier $\frac{1}{4}$ when $i = j = 0$, simultaneously. Now substituting the series (6) into equation (1), we have

$$
\sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' a_{ij}^{(4x)} T_i(x) T_j(y) + \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' a_{ij}^{(2x,2y)} T_i(x) T_j(y)
+ \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' a_{ij}^{(4y)} T_i(x) T_j(y) = - \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' g_{ij} T_i(x) T_j(y).
$$

Then, equating the coefficients of the same degrees of polynomials, we have an algebraic system

$$
\begin{gathered}
a_{ij}^{(4x)} + a_{ij}^{(2x,2y)} + a_{ij}^{(4y)} = -g_{ij}, \\
i = 0, 1, 2, ..., N, \\
j = 0, 1, 2, ..., M.
\end{gathered} \tag{7}
$$

The order of derivatives in system (7) can be reduced using the following discrete integration formulas:

$$
a_{ij}^{((k-1)x)} = \frac{a_{i-1,j}^{(kx)} - a_{i+1,j}^{(kx)}}{2i}, a_{ij}^{((k-1)y)} = \frac{a_{i,j-1}^{(ky)} - a_{i,j+1}^{(ky)}}{2j}. \tag{8}
$$

Using these formulas, system (7) is "integrated" four times with respect to the variable "x" and with respect to the variable "y". For example, at the first step of integration with respect to the variable "x" we have:

$$
a_{ij}^{(3x)} + 2a_{ij}^{(x,2y)} + \frac{a_{i-1,j}^{(4y)} - a_{i+1,j}^{(4y)}}{2i} = -\left(\frac{g_{i-1,j} - g_{i+1,j}}{2i} \right).
$$

After performing all integration operations and the requirement to satisfy the boundary conditions (2) for the approximate solution

$$u_a(x,y) = \sum_{i=0}^{N}{}' \sum_{j=0}^{M}{}' a_{ij} T_i(x) T_j(y) \tag{9}$$

leads to the following system of linear algebraic equations for determining the unknown expansion coefficients $a_{ij}(i = 0, 1, 2, ..., N, j = 0, 1, 2, ..., M)$:

$$\begin{cases} \left.\begin{array}{l} \frac{1}{2}a_{i,0} + a_{i,2} + a_{i,4} + \ldots + \\ +a_{i,2M} = 0, \\ a_{i,1} + a_{i,3} + a_{i,5} + \cdots + \\ +a_{i,2M-1} = 0, \\ 4a_{i,2} + 16a_{i,4} + 36a_{i,6} + \cdots + \\ +(2M)^2 a_{i,2M} = 0, \\ a_{i,1} + 9a_{i,3} + 25a_{i,5} + \cdots + \\ +(2M-1)^2 a_{i,2M-1} = 0, \end{array}\right\} (i = \overline{0, N}), \\ \\ \left.\begin{array}{l} \frac{1}{2}a_{0,j} + a_{2,j} + a_{4,j} + \cdots + \\ +a_{2N,j} = 0, \\ a_{1,j} + a_{3,j} + \cdots + a_{2N-1,j} = 0, \\ 4a_{2,j} + 16a_{4,j} + 36a_{6,j} + \cdots + \\ +(2N)^2 a_{2N,j} = 0, \end{array}\right\} (j = \overline{4, M}), \end{cases} \tag{10}$$

$$
\begin{cases}
a_{1,j} + 9a_{3,j} + 25a_{5,j} + \cdots + \\
+(2N-1)^2 a_{2N-1,j} = 0,\ (j = \overline{4, M}), \\
K_1(j^2-1)((j+1)(j+2)(j+3)a_{i,j-4} + (j-1)\times \\
\times(j-2)(j-3)a_{i,j+4}) - (K_1(j^2-1)(j+1)(j+2)\times \\
\times(j+3) + Z_1)a_{i,j-2} - (K_1(j^2-1)(j-1)(j-2)\times \\
\times(j-3) + Z_2)a_{i,j+2} + (Z_1+Z_2)a_{i,j} + K_2(j^2-9)\times \\
\times(i+1)(j+1)a_{i-2,j-2} - 2K_2 j(j^2-9)(i+1)a_{i-2,j}+ \\
+K_2(j^2-9)(i+1)(j-1)a_{i-2,j+2} + K_2(j^2-9)\times \\
\times(i-1)(j+1)a_{i+2,j-2} - 2K_2 j(j^2-9)(i-1)a_{i+2,j}+ \\
+K_2(j^2-9)(i-1)(j-1)a_{i+2,j+2} + 16j^2(j^2-1)^2\times \\
\times(j^2-4)(j^2-9)(C_1T_1 - C_2T_2 + C_3T_3)) = -C_1\times \\
\times(i-1)j((j+1)(j+2)(j+3)(j^2-1)g_{i-4,j-4}- \\
-4(j^2-1)(j^2-4)(j+3)g_{i-4,j-2} + 6j(j^2-1)\times \\
\times(j^2-9)g_{i-4,j} - 4(j^2-1)(j^2-4)(j-3)g_{i-4,j+2}+ \\
+(j-1)(j-2)(j-3)(j^2-1)g_{i-4,j+4})+ \\
+j(2c_1(i-2) + C_2(i+1))((j+1)(j+2)(j+3)\times \\
\times(j^2-1)g_{i-2,j-4} - 4(j^2-1)(j^2-4)(j+3)g_{i-2,j-2}+ \\
+6j(j^2-1)(j^2-9)g_{i-2,j} - 4(j^2-1)(j^2-4)\times \\
\times(j-3)g_{i-2,j+2} + (j-1)(j-2)(j-3)\cdot(j^2-1)\times \\
\times g_{i-2,j+4}) - j(C_1(i-3) + 2C_2 i + C_3(i+3))((j+1)\times \\
(j+2)(j+3)(j^2-1)g_{i,j-4} - 4(j^2-1)(j^2-4)\times \\
\times(j+3)g_{i,j-2} + 6j(j^2-1)(j^2-9)g_{i,j} - 4(j^2-1)\times \\
\times(j^2-4)(j-3)g_{i,j+2} + (j-1)(j-2)(j-3)\times \\
\times(j^2-1)g_{i,j+4}) + j(C_2(i-1) + 2C_3(i+2))\cdot((j+1)\times \\
\times(j+2)(j+3)(j^2-1)g_{i+2,j-4} - 4(j^2-1)(j^2-4)\times \\
\times(j+3)g_{i+2,j-2} + 6j(j^2-1)(j^2-9)g_{i+2,j}- \\
-4(j^2-1)(j^2-4)(j-3)g_{i+2,j+2} + (j-1)(j-2)\times \\
\times(j-3)(j^2-1)g_{i+2,j+4}) - C_3(i+1)j((j+1)(j+2)\times \\
\times(j+3)(j^2-1)g_{i+4,j-4} - 4(j^2-1)(j^2-4)(j+3)\times \\
\times g_{i+4,j-2} + 6j(j^2-1)(j^2-9)g_{i+4,j} - 4(j^2-1)\times \\
\times(j^2-4)(j-3)g_{i+4,j+2} + (j-1)\cdot(j-2)(j-3)\times \\
\times(j^2-1)g_{i+4,j+4}).
\end{cases}
\tag{10'}
$$

Here

$$
\begin{aligned}
K_1 &= 16i^2j(i^2-1)^2(i^2-4)(i^2-9),\\
K_2 &= 32ij(i^2-1)(i^2-4)(i^2-9)(j^2-1)(j^2-4),\\
T_1 &= (i-1)a_{i-4,j}-2(i-2)a_{i-2,j}+(i-3)a_{i,j},\\
T_2 &= (i+1)a_{i-2,j}-2ia_{i,j}+(i-1)a_{i+2,j},\\
T_3 &= (i+3)a_{i,j}-2(i+2)a_{i+2,j}+(i+1)a_{i+4,j},\\
Z_1 &= (3K_1(j-1)(j+2)+2K_2i)(j^2-9)(j+1),\\
Z_2 &= (3K_1(j-2)(j+1)+2K_2i)(j^2-9)(j-1),\\
C_1 &= i(i+1)^2(i+2)(i+3),\\
C_2 &= 2i(i^2-4)(i^2-9),\\
C_3 &= i(i-1)^2(i-2)(i-3).
\end{aligned}
$$

To determine the right-hand side g_{ij} of equation (7) from the known function $f(x,y)$in the collocation nodes of the Chebyshev polynomials $x_l = \cos\frac{\pi l}{N}(l = 0, 1, ..., N)$, $y_k = \cos\frac{\pi k}{M}(k = 0, 1, ..., M)$ there is a discrete inverse transformation [14-15]:

$$
g_{ij} = \frac{4}{MNc_ic_j}\sum_{l=0}^{N}\sum_{k=0}^{M}\left(\frac{1}{c_lc_k}f(x_l,y_k)T_i(x_l)T_j(y_k)\right),
\quad i = 0, 1, ..., N, j = 0, 1, ..., M, \tag{11}
$$

where $c_0 = c_N = c_M = 2, c_m = 1$ at $m = 1, 2, ..., N-1, c_t = 1$ at $t = 1, 2, ..., M-1$. It is convenient to write the systems (10) in matrix form

$$
Ax = b, \tag{12}
$$

where is Aa square matrix of order $K \times K$, here $K = (N+1) \times$ $\times (M+1)$ consisting of the coefficients of systems (10),

$$
x^T = (a_{00}, a_{10}, ..., a_{N0}, a_{01}, a_{11}, ..., a_{N1}, ..., a_{0M}, a_{1M}, ..., a_{NM})
$$

is the desired vector for the unknown expansion coefficients, b is the right-hand side of systems (10). Solving system (10), the coefficients are determined, then the values of the exact solution are calculated using formulas (4), and the values of the approximate solution in $a_{ij}(i = 0, 1, ..., N; j = 0, 1, ..., M)$ the collocation nodes of the Chebyshev polynomials (x_l, y_k) are calculated using formula (9).

4. **Calculation results**

Let us present the results of numerical calculations solution of the boundary value problem for the biharmonic equation (1) - (2), using the above-described discrete version of the preliminary integration method.

Table 1 shows the results of comparison of the exact and approximate solutions for the selected trial functions. $u_e^{(1)}(x,y)$, $u_e^{(2)}(x,y)$ determined by formula (4) in the case when the exponent of the exponential function is $A = 1, 3$, and the number of approximating Chebyshev polynomials both for the variable "x" and for the variable "y " is equal to 30, i.e. $N = M = 30$. It is evident that with the selected values of the characteristic parameters, approximate solutions to problem (1) - (2) $u_a^{(1)}(x,y)$ are $u_a^{(2)}(x,y)$ found with sufficiently high accuracy, while the absolute error in specific collocation nodes is a value of the order of 10^{-14}.

Table 1. Comparison of absolute errors for exact and approximate solutions

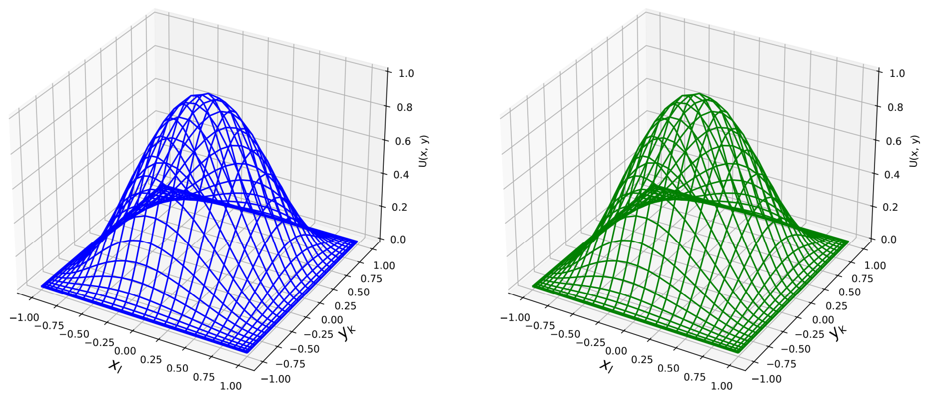
Calculation results for the exact solution $u_e^{(1)}(x,y)$					
A	x_l $l=\overline{0,N}$ at l	y_k $k=\overline{0,M}$ at k	Values of the exact solution	Approximate solution values	Absolute error
1	10	10	0.00010037975586234	0.00010037975586233	$7.94 \cdot 10^{-18}$
	20	20	0.4736033572362967	0.4736033572362972	$4.99 \cdot 10^{-16}$
3	10	10	0.0002242935716098	0.0002242935716095	$3.09 \cdot 10^{-16}$
	20	20	0.69208265178736	0.69208265178737	$6.77 \cdot 10^{-15}$
Calculation results for the exact solution $u_e^{(2)}(x,y)$					
1	10	10	0.000299311498722039	0.000299311498722043	$4.34 \cdot 10^{-18}$
	20	20	0.970138852140147	0.970138852140146	$4.44 \cdot 10^{-16}$
3	10	10	0.00594632477949	0.00594632477947	$2.05 \cdot 10^{-14}$
	20	20	5.94860783552934	5.94860783552933	$1.24 \cdot 10^{-14}$

The results of Table 1 are most clearly illustrated in Fig. 1-4. Fig. 1 shows the graphs of the exact and approximate solution for the selected trial function $u_e^{(1)}(x,y)$, when the values of the parameter $A = 1$ and the number of polynomials are equal to $N = M = 30$.

The graphs of the exact and approximate solutions for the function are $u_e^{(1)}(x,y)$ shown $A = 3, N = M = 30$ in Fig. 2.

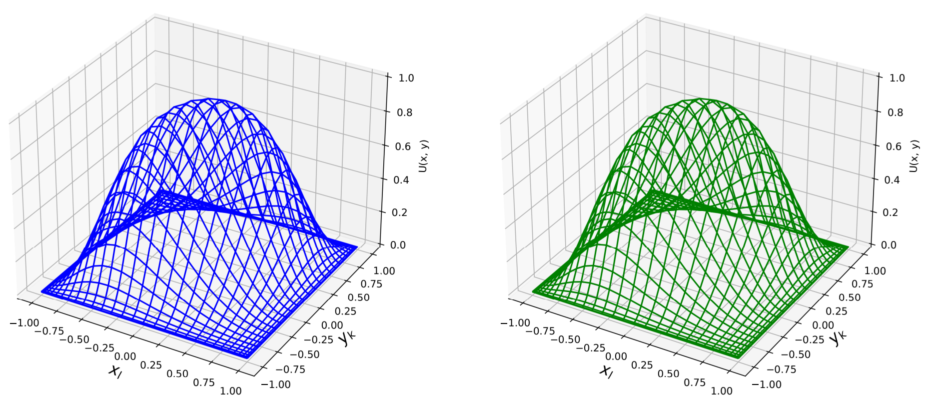
Fig. 3 shows graphs of the exact and approximate solution for the function $u_e^{(2)}(x,y)$ with the following values of characteristic parameters: $A = 1, N = M = 30$.

The dynamics of the exact and approximate solutions for the function $u_e^{(2)}(x,y)$ are $A = 3, N = M = 30$ shown in Fig. 4.



a) exact solution $u_e^{(1)}$ **b)** approximate solution $u_a^{(1)}$

FIGURE 1. Graphs of the exact and approximate solution for the function $u_e^{(1)}(x, y)$ at $A = 1$.



a) exact solution $u_e^{(1)}$ **b)** approximate solution $u_a^{(1)}$

FIGURE 2. Graphs of the exact and approximate solution for the function $u_e^{(1)}(x, y)$ at $A = 3$.

Table 2 shows the maximum absolute errors for the selected trial functions (4) in the collocation nodes of the Chebyshev polynomials (x_l, y_k) for the following values of the characteristic parameters: $A = 1, 3, N = M = 5 \div 30$. The maximum absolute error is determined by the following

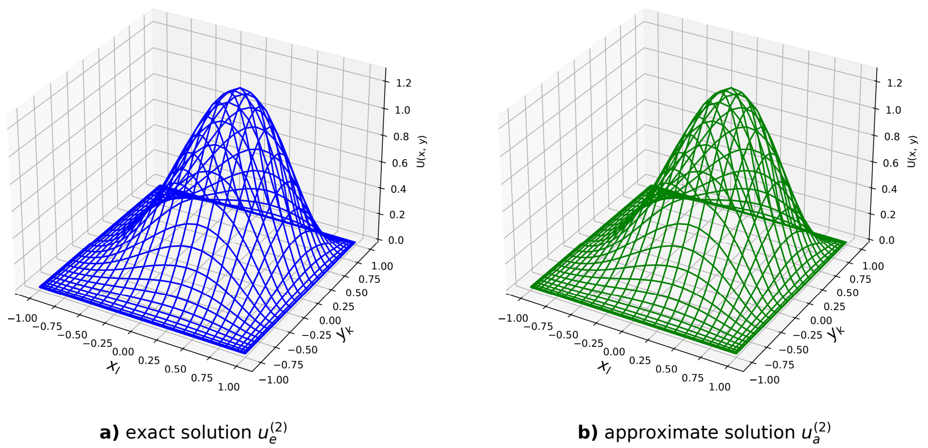


Figure 3. Graphs of the exact and approximate solution for the function $u_e^{(2)}(x, y)$ at $A = 1$.

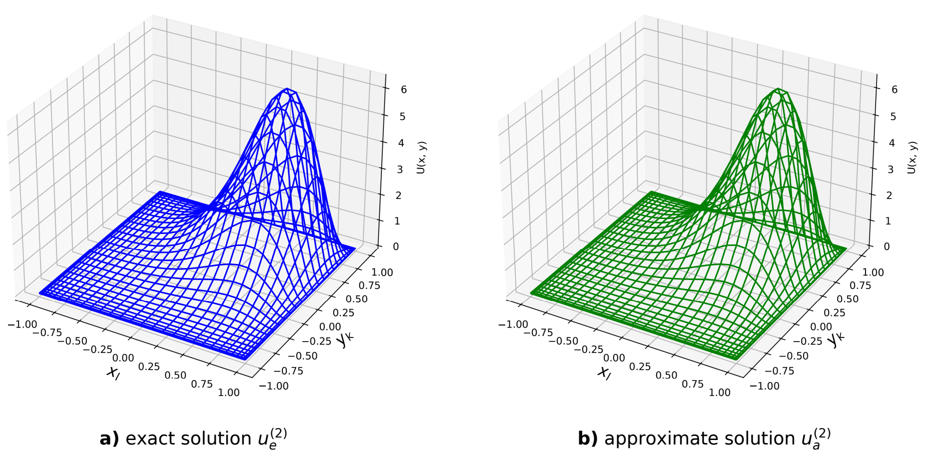


Figure 4. Graphs of the exact and approximate solution for the function $u_e^{(2)}(x, y)$ at $A = 3$.

formula

$$\Delta = \max_{\substack{0 \le l \le N \\ 0 \le k \le M}} \left| u_e^{(p)}(x_l, y_k) - u_a^{(p)}(x_l, y_k) \right|, p = 1, 2.$$

Table 2. Maximum absolute errors of specific nodes for different values of characteristic parameters A, N, M.

A	N	M	x_l $l=\overline{0,N}$ at l	y_k $k=\overline{0,M}$ at k	Trial function $u_e^{(1)}(x,y)$	x_l $l=\overline{0,N}$ at l	y_k $k=\overline{0,M}$ at k	Trial function $u_e^{(2)}(x,y)$
1	5	5	2	3	0.01	2	2	0.25
	10	10	3	7	$1.21 \cdot 10^{-5}$	3	4	$5.24 \cdot 10^{-6}$
	15	15	11	11	$4.81 \cdot 10^{-9}$	7	7	$6.58 \cdot 10^{-12}$
	20	20	15	15	$5.92 \cdot 10^{-13}$	8	10	$2.67 \cdot 10^{-15}$
	25	25	15	10	$1.67 \cdot 10^{-15}$	12	12	$3.78 \cdot 10^{-15}$
	30	30	19	14	$1.78 \cdot 10^{-15}$	14	12	$3.55 \cdot 10^{-15}$
3	5	5	2	3	0.05	1	2	2.36
	10	10	7	7	$0.03 \cdot 10^{-2}$	3	3	0.03
	15	15	11	4	$4.31 \cdot 10^{-7}$	6	6	$6.09 \cdot 10^{-6}$
	20	20	5	5	$3.98 \cdot 10^{-10}$	6	6	$8.19 \cdot 10^{-11}$
	25	25	6	6	$1.11 \cdot 10^{-13}$	25	12	$6.16 \cdot 10^{-13}$
	30	30	8	14	$1.03 \cdot 10^{-14}$	20	15	$4.11 \cdot 10^{-13}$

From the results presented in Table 2 it is evident that with a gradual increase in the number of approximating Chebyshev polynomials with a step equal to 5, the maximum absolute error for both trial functions decreases at a geometric progression rate.

5. **Conclusion**

For the numerical solution of the biharmonic equation, a highly accurate and efficient method is proposed - a discrete version of the preliminary integration method.

An algorithm for solving the proposed method has been developed. A large-scale computational experiment has been conducted with different values characteristic parameters and for different selected trial functions shows that the maximum absolute error decreases at a geometric progression rate with an increase in the number of approximating Chebyshev polynomials.

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