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REFINED STABILITY AND CONVERGENCE FOR INVERSE PARABOLIC CAUCHY PROBLEMS VIA SPECTRAL REGULAR CONTROL

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Abstract

In this paper, we address the inverse problem of reconstructing the initial condition for a non-homogeneous parabolic equation from a prescribed final state. This inverse Cauchy problem is well known to be severely ill-posed, with solutions that may not exist or fail to depend continuously on the final data. We propose a novel regularization approach that introduces a controlled perturbation of the final value problem to overcome these difficulties. The proposed method yields significantly improved stability and convergence rates, surpassing classical Hölder-type and logarithmic-type estimates presented in previous studies. In particular, we derive new sharp uniform bounds and error estimates, highlighting the effectiveness of our approach in enhancing the stability of the inversion process.

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1. **Introduction**

Inverse parabolic Cauchy problems have attracted considerable research interest since the 1960s due to their relevance in a wide range of physical applications, particularly in time-reversed heat transfer processes (see [9],[10],[12],[16]). These problems are inherently ill-posed: small perturbations in the final data can lead to significant deviations in the reconstructed initial state or even the' non-existence of solutions. Several regularization techniques have been developed to address this challenge to ensure stability and enable meaningful approximations of the ill-posed models. Notable approaches include the quasi-reversibility method introduced by Lattès and Lions [9], the method of auxiliary boundary conditions [8, 12], the quasi-boundary value method [1, 3, 11, 21], and the C-regularized semigroup framework [12, 13].

In this work, we focus on the following inverse control problem: given a final state $g \in \mathcal{H}$, determine the initial condition $h \in \mathcal{H}$ such that the solution u of the non-homogeneous forward Cauchy problem

$$\begin{cases} u_t(t) + Au(t) = f, & 0 < t < T, \\ u(0) = h, \end{cases} \tag{IVP}$$

satisfies the final condition $u(T) = g$, A denotes a self-adjoint, positive, unbounded linear operator defined on a Hilbert space $\mathcal{H}$. A natural approach is to recast this problem as a final value problem and seek a solution u satisfying:

$$\begin{cases} u_t(t) + Au(t) = f, & 0 < t < T, \\ u(T) = g. \end{cases} \tag{FVP}$$

However, this formulation is severely ill-posed and requires suitable regularization techniques to obtain a stable solution.

The inverse final value parabolic problem (FVP) is ill-posed: solutions may not exist, and when they do, they often depend sensitively on various problem parameters. Furthermore, such solutions typically lack continuous dependence on the final data, making the problem highly unstable. In this context, we propose a novel regularization framework that enables effective control over stability and the approximation error. Our method offers a robust alternative for addressing the instability inherent in the inverse formulation and improves upon existing regularization strategies.

Since the work of Lattès and Lions [9] on the quasi-reversibility method, various approaches have been developed to deal with the ill-posedness of the FVP. These include the stabilized approaches introduced by Miller [14], the quasi-boundary value method developed by Clark and Oppenheimer [1], Denche and Bessila [3]. The approach by auxiliary condition techniques formulated by Ivanov, Melnikova, and Filinkov [8, 12]. In this wide range of literature on the (FVP)-Problem, chronological improvements on the stability and error estimates have been treated. In the works [4, 5] stability was of order $e^{\frac{T}{\alpha}}$, in [1, 17, 7, 21] a better estimate of order $\alpha^{\frac{t}{T}-1}$ is given, afterwards in [3, 18] the stability order is improved to $\frac{1}{\alpha \log \frac{1}{\alpha}}$; then the better order $\frac{1}{\alpha\left(\alpha \log \frac{1}{\alpha}\right)^p}$, $p \geq 1$ is given in [2], finally for the inverse source heat equation [19] the stability is improved to the Hölder type $\left(\frac{1}{\alpha}\right)^{\frac{1}{s}}$, $s \geq 1$.

In this paper, we introduce a novel perturbation framework inspired by the C-regularized semigroup theory, originally discussed by Melnikova and Filinkov in [12, 13]. The proposed method enables us to derive new stability bounds that significantly outperform classical Hölder-type estimates of the form $\alpha^{-\frac{1}{s}}$, with $s \geq 1$. We establish uniform convergence rates and error estimates that go beyond the traditional Hölderian and logarithmic regularization results. These findings represent a substantial improvement over those reported in several recent studies addressing inverse heat equations and backward parabolic problems, including [1, 2, 3, 7, 11, 15, 18, 19, 20, 21]. For this purpose, we introduce the following sequence of problems to approximate the non-homogeneous inverse in time Cauchy problem (FVP):

$$\begin{cases} u_t(t) + Au(t) = f_\alpha, & 0 < t < T, \\ u(T) = g_\alpha, \end{cases}$$

where f_α and g_α are suitably perturbed versions of the original data. In this study, we establish the well-posedness and uniqueness of classical solutions for the proposed approximate problems. Building upon this framework, we derive novel sharp uniform stability bounds and uniform error estimates, thereby confirming the effectiveness of the proposed regularization method.

2. The approximate problem

We revisit the inverse problem of determining the initial state $u \in \mathcal{H}$ such that the solution u of the non-homogeneous parabolic Cauchy problem

$$\begin{cases} u_t(t) + Au(t) = f, & 0 < t < T, \\ u(0) = h, \end{cases} \tag{IVP}$$

satisfies the prescribed final condition $u(T) = g$, where $g \in \mathcal{H}$ and $f \in L^2((0,T), \mathcal{H})$. The operator A is assumed to be self-adjoint, positive, and unbounded on the Hilbert space $\mathcal{H}$. It admits a complete orthonormal system of eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}^*}$ in $\mathcal{H}$, associated one-to-one with a strictly increasing, unbounded sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^*}$. An obvious solution to this problem is to find a solution u of the non-homogeneous inverse in time Cauchy problem:

$$\begin{cases} u_t(t) + Au(t) = f, & 0 < t < T, \\ u(T) = g, \end{cases} \tag{FVP}$$

where $g \in \mathcal{H}$ and $f \in L^2((0,T), \mathcal{H})$. However, it is well known that this problem is severely ill-posed in the sense of Hadamard: small perturbations in the final data g may lead to large deviations in the reconstructed solution $u(0)$.

To regularize the instability of the inverse problem, we propose a modified quasi-boundary regularization method based on a regularization parameter α. The idea is to perturb both the final condition g and the source term f, leading to the following regularized problem

$$\begin{cases} u_t(t) + Au(t) = f_\alpha, & 0 < t < T, \\ u(T) = g_\alpha, \end{cases} \tag{AFVP}$$

where we define for each $0 < \alpha < 1$:

$$f_\alpha = \sum_{k\geq 1} \frac{1}{\alpha e^{\lambda_k^p T} + 1} f_k \varphi_k \quad \text{and} \quad g_\alpha = \sum_{k\geq 1} \frac{1}{\alpha e^{\lambda_k^p T} + 1} g_k \varphi_k. \tag{1}$$

The $\{f_k\}_{k\in\mathbb{N}^*}$ and $\{g_k\}_{k\in\mathbb{N}^*}$ are the sequences of the Fourier-Bessel Coefficients of f and g in the Hilbert space $\mathcal{H}$, and $\{\lambda_k\}_{k\in\mathbb{N}^*}$ is the sequence of eigenvalues of the operator A. The real parameter $p > 1$ is arbitrary.

LEMMA 2.1. *For $g \in H$ and $f \in L^2((0,T),\mathcal{H})$, the (FVP) problem admits a solution in the classical sense if and only if*

$$\sum_{k\geq 1}\left(g_k e^{\lambda_k T} - \int_t^T e^{\lambda_k \tau} f_k(\tau)d\tau\right)^2 < +\infty. \tag{2}$$

P r o o f. Assume that the(FVP) problem admits a classical solution. This implies:

$$u(t) = \sum_{k\geq 1} u_k(t)\varphi_k\ ,\ u(t) \in \mathcal{D}(A)\ , \forall t \in (0,T) \text{ and } u \in C^1((0,T),\mathcal{H}).$$

From the equation and the final value condition in the (FVP) problem, we infer that

$$u\left(t\right) = \sum_{k\geq 1}\left(g_k e^{\lambda_k (T-t)} - \int_t^T e^{\lambda_k (T-\tau)} f_k\left(\tau\right)d\tau\right)\varphi_k, \tag{3}$$

where f_k and g_k are the Fourier-Bessel coefficients of f and g in the space $\mathcal{H}$. Since $u(t) \in \mathcal{H}$ for all $t \in [0,T]$, therefore:

$$\left\|u\left(t\right)\right\|^2 = \sum_{k\geq 1}\left(g_k e^{\lambda_k (T-t)} - \int_t^T e^{\lambda_k (T-\tau)} f_k\left(\tau\right)d\tau\right)^2 < +\infty,$$

therefore for $t = 0$, we obtain

$$\left\|u\left(0\right)\right\|^2 = \sum_{k\geq 1}\left(g_k e^{\lambda_k T} - \int_0^T e^{\lambda_k \tau} f_k\left(\tau\right)d\tau\right)^2 < +\infty.$$

Conversely, assume that (2) is satisfied. Then we introduce the function $u(t)$ written in expression (3), from Cauchy-Schwartz and direct inequalities, we get

$$\left\|u\left(t\right)\right\|^2 \leq 2\sum_{k\geq 1}\left(g_k e^{\lambda_k T} - \int_0^T e^{\lambda_k \tau} f_k\left(\tau\right)d\tau\right)^2 + 2T\left\|f\right\|^2_{L^2((0,T),\mathcal{H})}.$$

Similarly, we have for $Au(t)$

$$\left\|Au\left(t\right)\right\|^2 \leq \sum_{k\geq 1}\lambda_k^2 e^{-2\lambda_k t}\left(g_k e^{\lambda_k T} - \int_t^T e^{\lambda_k \tau} f_k\left(\tau\right)d\tau\right)^2,$$

which implies

$$\|Au(t)\|^2 \leq 2\sum_{k\geq 1}\left(g_k e^{\lambda_k T} - \int_0^T e^{\lambda_k \tau} f_k(\tau)\, d\tau\right)^2 + 2T\,\|f\|^2_{L^2((0,T),\mathcal{H})},$$

therefore, $u(t) \in \mathcal{D}(A)$, $\forall t \in (0,T)$. Finally, because we have:

$$u_t(t) = \sum_{k\geq 1}\left[f_k(t) - \lambda_k e^{-\lambda_k t}\left(g_k e^{\lambda_k T} - \int_t^T e^{\lambda_k \tau} f_k(\tau)\, d\tau\right)\right]\varphi_k,$$

we infer that $u \in C^1((0,T),\mathcal{H})$, we check that $u(t)$ verifies well the (FVP) model which ends the proof. □

3. Stability, convergence and error-estimates

In this section, various new stability, convergence, and error estimates are established for the proposed regularization method.

THEOREM 3.1. *If $g \in H$ and $f \in L^2((0,T),\mathcal{H})$ the approximate non-homogeneous inverse in time Cauchy process (AFVP) admits a unique classical solution $u_\alpha(t)$, depending continuously on both data g in $\mathcal{H}$ and f in $L^2((0,T),\mathcal{H})$. Moreover, for each $t \in [0,T]$ and $p > 1$ we have*

$$\|u_\alpha(t)\| \leq e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}\left(\|g\| + \sqrt{T}\,\|f\|_{L^2((0,T),\mathcal{H})}\right). \tag{4}$$

P r o o f. In the approximating process (AFVP), if $g \in H$ and $f \in L^2((0,T),\mathcal{H})$ a unique solution does exist by the formal expression,

$$u_\alpha(t) = \sum_{k\geq 1}\frac{e^{\lambda_k(T-t)}}{\alpha e^{\lambda_k^p T} + 1}\left(g_k - \int_t^T e^{\lambda_k(\tau - T)} f_k(\tau)\, d\tau\right)\varphi_k, \tag{5}$$

then

$$\|u_\alpha(t)\| \leq \left\|\sum_{k\geq 1}\frac{e^{\lambda_k(T-t)}}{\alpha e^{\lambda_k^p T}+1} g_k\varphi_k\right\| + \left\|\sum_{k\geq 1}\frac{e^{\lambda_k(T-t)}}{\alpha e^{\lambda_k^p T}+1}\int_t^T e^{\lambda_k(\tau-T)} f_k(\tau)\, d\tau\varphi_k\right\|. \tag{6}$$

If we take the function

$$h_1(\lambda) = \frac{e^{\lambda T}}{\alpha e^{\lambda^p T} + 1},\ \lambda \geq 0,$$

then $h_1(\lambda)$ achieves its supremum at the critical value λ_0 solution of the equation,

$$\alpha e^{\lambda^p T}\left(p\lambda^{p-1}-1\right)=1,\ p>1,$$

which we may write as,

$$\lambda^p T\left[1+\frac{\log\left(p\lambda^{p-1}\right)}{\lambda^p T}+\frac{1}{\lambda^p T}\log\left(1-\frac{1}{p\lambda^{p-1}}\right)\right]=\log\frac{1}{\alpha},$$

for small enough values of α , we may choose,

$$\lambda_0^p T\approx\log\frac{1}{\alpha}\Leftrightarrow\lambda_0\approx\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}},$$

which implies that,

$$\underset{\lambda\geq 0}{\sup} h_1(\lambda)\leq h_1(\lambda_0)=\frac{e^{\lambda_0 T}}{\alpha\frac{1}{\alpha\left(p\lambda_0^{p-1}-1\right)}+1}\leq e^{\lambda_0 T},$$

with

$$e^{\lambda_0 T}=e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}},\ p>1.$$

Hence, a substitution in the inequality (6) and a Cauchy-Schwartz inequality, imply for each $p>1$,

$$\left\|u_\alpha(t)\right\|\leq e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}\left(\|g\|+\sqrt{T}\,\|f\|_{L^2((0,T),\mathcal{H})}\right),$$

therefore $u_\alpha(t)\in\mathcal{H}$, $\forall t\in[0,T]$. In the same way as in the proof of Lemma 2.1 , we show that $u_\alpha(t)\in\mathcal{D}(A)$,$\forall t\in(0,T)$ and $u_\alpha\in C^1((0,T),\mathcal{H})$, which completes the proof. □

According to the final data g in $\mathcal{H}$, we may deduce the stability in the corollary below.

Corollary 3.1. *Let g_1 and g_2 be final values in $\mathcal{H}$ associated with the solutions $u_{1\alpha}(t)$ and $u_{2\alpha}(t)$, then*

$$\left\|u_{1\alpha}(t)-u_{2\alpha}(t)\right\|\leq e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}\left\|g_1-g_2\right\|,\tag{7}$$

for each $p>1$.

P r o o f. Respectively from solutions $u_\alpha(t)$ in formula (5), and the supremum inequality in theorem 3.1, we get

$$\begin{aligned}\|u_{1\alpha}(t)-u_{2\alpha}(t)\| &= \sum_{k\geq 1}\frac{e^{2\lambda_k(T-t)}}{\left(\alpha e^{\lambda_k^p T}+1\right)^2}\left(g_{1ki}-g_{2ki}\right)\\ &\leq e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}\|g_1-g_2\|,\end{aligned}$$

for each $p>1$. □

REMARK 3.1. In the control process of the final value inverse Cauchy problem (FVP), according to Theorem 3.1, we have a sharp norm estimate and stability of order $e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}$, $p>1$. Furthermore, $e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}$ expresses a stability finer than that of Hölderian type $\alpha^{-\frac{1}{s}}$, $s>1$ and than the logarithmic stability of order $\frac{1}{\alpha\log\frac{T}{\alpha}}$ and $\frac{1}{\alpha\left(\log\frac{T}{\alpha}\right)^p}$, since we have

$$e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}<\left(\frac{1}{\alpha}\right)^{\frac{1}{s}}\Longleftrightarrow\frac{1}{T}\log\frac{1}{\alpha}<\left(\frac{1}{sT}\log\frac{1}{\alpha}\right)^p,\ \forall p>1,\forall s\geq 1.$$

For the isolated case $p=1$, Theorem 3.1 reduces to,

$$\|u_\alpha(t)\|\leq\frac{1}{\alpha}\left(\|g\|_{\mathcal{H}}+\sqrt{T}\,\|f\|_{L^2((0,T),\mathcal{H})}\right). \tag{8}$$

Concerning the variable $t\in[0,T]$ the previous uniform norm estimate may be made sharp as follows.

THEOREM 3.2. *Let $g\in\mathcal{H}$ and $f\in L^2((0,T),\mathcal{H})$, then for each $p>1$, $0<\alpha<1$ and $t\in[0,T]$, we have*

$$\|u_\alpha(t)\|\leq e^{(T-t)\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}\left(\|g\|+\sqrt{T-t}\,\|f\|_{L^2((0,T),\mathcal{H})}\right). \tag{9}$$

P r o o f. Using

$$\begin{aligned}\frac{e^{2\lambda_k(T-t)}}{\left(\alpha e^{\lambda_k^p T}+1\right)^2} &\leq \frac{e^{2\lambda_k(T-t)}}{\left[\left(\alpha e^{\lambda_k^p T}+1\right)^{\frac{t}{T}}\left(\alpha e^{\lambda_k^p T}+1\right)^{1-\frac{t}{T}}\right]^2}\\ &\leq\left(\frac{e^{2\lambda_k(T-t)}}{\alpha e^{\lambda_k^p T}+1}\right)^{2\left(1-\frac{t}{T}\right)},\end{aligned}$$

from the sup-estimate as in Theorem 3.1, we get

$$\left\| \sum_{k\geq 1} \frac{e^{\lambda_k (T-t)}}{\alpha e^{\lambda_k^p T}+1} g_k \varphi_k \right\| \leq e^{(T-t)\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}} \|g\|,$$

similarly, we have,

$$\left\| \sum_{k\geq 1} \frac{e^{\lambda_k (T-t)}}{\alpha e^{\lambda_k^p T}+1} \int_0^T e^{\lambda_k(\tau - t)} f_k(\tau)\, d\tau \varphi_k \right\|$$

$$\leq e^{(T-t)\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}} \sqrt{(T-t)}\, \|f\|_{L^2((0,T),\mathcal{H})},$$

replacing into inequality (6), we get the required result. □

Theorem 3.3. *For all $g \in \mathcal{H}$, as α tends to zero, the sequence $(u_\alpha(T))$ converges to g in $\mathcal{H}$.*

P r o o f. For any $0 < \alpha < 1$ and $p > 1$, from (5) we have

$$\|u_\alpha(T) - g\|^2 = \sum_{k\geq 1} \left[\frac{\alpha e^{\lambda_k^p T}}{\alpha e^{\lambda_k^p T}+1}\right]^2 g_k^2$$

then, for $\varepsilon > 0$, let $N \in \mathbb{N}$ such that $\sum\limits_{k\geq N+1} g_k^2 \leq \frac{\varepsilon}{2}$, hence

$$\|u_\alpha(T) - g\|^2 = \sum_{k=1}^{N} \left[\frac{\alpha e^{\lambda_k^p T}}{\alpha e^{\lambda_k^p T}+1}\right]^2 g_k^2 + \sum_{N+1}^{\infty} \left[\frac{\alpha e^{\lambda_k^p T}}{\alpha e^{\lambda_k^p T}+1}\right]^2 g_k^2,$$

this implies

$$\|u_\alpha(T) - g\|^2 = \alpha^2 \sum_{k=1}^{N} e^{2\lambda_k^p T} g_k^2 + \frac{\varepsilon}{2},$$

if we choose $\alpha < \sqrt{\frac{\varepsilon}{2}} \left(\sum_{k=1}^N e^{2\lambda_k^p T} g_k^2\right)^{-\frac{1}{2}}$, then $\|u_\alpha(T) - g\|^2 \leq \varepsilon$, this ends the proof. □

In the theorems below, we establish the new error estimates on the convergence rates of the sequence $u_\alpha(T)$ to the final data g in the Hilbert space $\mathcal{H}$. These theorems express various results according to different conditions about the variations of $g \in \mathcal{H}$, later on, we shall rely on these latter to pursue generalisations to the uniform behaviour on the interval $[0, T]$.

Theorem 3.4. *Let us consider $g \in \mathcal{H}$, assume that for some $r > 0$ and $\epsilon > 0$ with $\epsilon \leq p$ the series $\sum\limits_{k\geq 1} e^{2r\lambda_k^\epsilon T} g_k^2$ converges, then there exists a constant $c > 0$ which depends on $g \in \mathcal{H}$ for which,*

$$\|u_\alpha(T) - g\| \leq ce^{-rT\left(\frac{1}{T}\log\frac{1}{a}\right)^{\frac{\epsilon}{p}}}. \tag{10}$$

P r o o f. Since

$$\|u_\alpha(T) - g\|^2 = \sum_{k\geq 1}\left[\frac{\alpha e^{\left(\lambda_k^p - r\lambda_k^\epsilon\right)T}}{\alpha e^{\lambda_k^p T} + 1}\right]^2 e^{2r\lambda_k^\epsilon T} g_k^2,$$

set the function

$$h_2(\lambda) = \frac{e^{(\lambda^p - r\lambda^\epsilon)}T}{\alpha e^{\lambda^p T} + 1},$$

then the supremum of $h_2(\lambda)$ is reached at the critical value λ_0 solution of the equation,

$$\alpha e^{\lambda^p T} = \left(\frac{p}{r\varepsilon}\lambda^{p-\epsilon} - 1\right), \tag{11}$$

which may be written as,

$$\lambda^p T\left[1 - \frac{1}{\lambda^p T}\log\left(\frac{p}{rc}\lambda^{p-\epsilon}\right) - \lambda^p T\log\left(1 - \frac{r\epsilon}{p\lambda^{p-\epsilon}}\right)\right] = \log\frac{1}{\alpha}.$$

For small values of α we may choose

$$\lambda_0^p T \approx \log\frac{1}{\alpha} \Longleftrightarrow \lambda_0 \approx \left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}.$$

Hence,

$$\sup_{\lambda\geq 0}\alpha h_2(\lambda) \leq \alpha h_2(\lambda_0) = \frac{\alpha e^{\left(\lambda_0^P - r\lambda_0^\epsilon\right)T}}{\alpha e^{\lambda_o^p T} + 1} \leq e^{-r\lambda_0^\epsilon T}, \tag{12}$$

since

$$\|u_\alpha(T) - g\|^2 = \sum_{k\geq 1}\left[\frac{\alpha e^{\left(\lambda_k^p - r\lambda_k^\epsilon\right)T}}{\alpha e^{\lambda_k^p T} + 1}\right]^2 e^{2r\lambda_k^\epsilon T} g_k^2,$$

therefore,

$$\|u_\alpha(T) - g\|^2 \leq c^2 e^{-2rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{\epsilon}{p}}},$$

finally:

$$\|u_\alpha(T) - g\| \leq ce^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{\epsilon}{p}}},$$

the constant c depends on $\sum\limits_{k\geq 1} e^{2r\lambda_k^\epsilon T} g_k^2$, $r > 0$ and $p \geq \epsilon > 0$. □

Various discussions on the above result are deduced according to the behaviour of the parameters ϵ and r, leading to new and known results in several recent papers. In this section, we prefer to state and prove the main results and leave the consequences and comparisons in the next section.

THEOREM 3.5. *Let the data $g \in \mathcal{H}$, assume that for some $r > 0$ the series $\sum_{k\geq 1} e^{2r\lambda_k T} g_k^2$ converges, there exists a constant $c > 0$ which depends on $g \in H$ for which*

$$\|u_\alpha(T) - g\| \leq c e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}. \tag{13}$$

P r o o f. Following steps as in Theorem 3.5, we have,

$$\begin{aligned}\|u_\alpha(T) - g\|^2 &= \sum_{k\geq 1}\left[\frac{\alpha e^{\lambda_k^p T}}{\alpha e^{\lambda_k^p T}+1}\right]^2 g_k^2 \\ &= \sum_{k\geq 1}\left(\frac{\alpha e^{\left(\lambda_k^p - r\lambda_k\right)T}}{\alpha e^{\lambda_k^p T}+1}\right)^2 e^{2r\lambda_k T} g_k^2,\end{aligned}$$

the function

$$h_3(\lambda) = \frac{e^{(\lambda^p - r\lambda)T}}{\alpha e^{\lambda^p T}+1},$$

reaches its supremum at the critical value λ_0 solution of the equation,

$$\alpha e^{\lambda^p T} = \left(\frac{p}{r}\lambda^{p-1} - 1\right),$$

we may write it as,

$$\lambda^p T\left[1 - \frac{1}{\lambda^P T}\log\left(\frac{p}{r}\lambda^{p-1}\right) - \frac{1}{\lambda^P T}\log\left(1 - \frac{r}{p\lambda^{p-1}}\right)\right] = \log\frac{1}{\alpha}.$$

Again for α small enough, we have

$$\lambda_0^p T \approx \log\frac{1}{\alpha} \Leftrightarrow \lambda_0 \approx \left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}},$$

thus,

$$\sup_{\lambda\geq 0}\alpha h_3(\lambda) = \alpha h_3(\lambda_0) = \frac{\alpha e^{\left(\lambda_0^p - r\lambda_0\right)T}}{\alpha e^{\lambda_0^p T}+1} \leq e^{-r\lambda_0 T},$$

therefore,

$$\|u_\alpha(T) - g\|^2 \leq c^2 e^{-2rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{P}}},$$

where c is a constant which depends on $\sum\limits_{k\geq 1} e^{2r\lambda_k T} g_k^2$, $r>0$, this finishes the proof. □

THEOREM 3.6. *Let the data $g\in\mathcal{H}$, assume that for some $s>0$ the series $\sum\limits_{k\geq 1}\lambda^{2s}g_k^2$ converges, there exists a constant $c>0$ which depends uniquely on $g\in\mathcal{H}$ for which*

$$\|u_\alpha(T)-g\|\leq c\frac{1}{\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{s}{p}}},\ \ s>0. \tag{14}$$

P r o o f. Performing similar steps, we have

$$\begin{aligned}\|u_\alpha(T)-g\|^2 &= \sum_{k\geq 1}\left(\frac{\alpha e^{\lambda_k^p T}}{\alpha e^{\lambda_k^p T}+1}\right) g_k^2\\ &= \sum_{k\geq 1}\left(\frac{\alpha e^{\lambda_k^p T}}{\lambda^s\left(\alpha e^{\lambda_k^p T}+1\right)}\right)^2 \lambda^{2s} g_k^2,\end{aligned}$$

we let the function

$$h_4(\lambda)=\frac{e^{\lambda^p T}}{\alpha\lambda^s e^{\lambda^p T}+1},$$

the supremum is reached at the critical value λ_0 solution of the equation,

$$\alpha e^{\lambda^p T}=\frac{pT}{s}\lambda^{p-s}.$$

For α small enough, we have the same behaviour,

$$\lambda_0^p T\approx\log\frac{1}{\alpha}\Leftrightarrow\lambda_0\approx\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}},$$

hence,

$$\sup_{\lambda\geq 0}\alpha h_4(\lambda)\leq\alpha h_4(\lambda_0)=\frac{\alpha e^{\lambda_0^p T}}{\alpha\lambda_0^s e^{\lambda_0^p T}+1}\leq\frac{1}{\lambda_0^s}, \tag{15}$$

therefore,

$$\|u_\alpha(T)-g\|^2\leq c^2\frac{1}{\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{2s}{p}}},$$

where c is a constant depending upon $\sum\limits_{k\geq 1}\lambda^{2s}g_k^2$, the parameters p and s are arbitrary, satisfying $p>1$ and $s>0$. □

The previous results give different error estimates corresponding respectively to different strong hyperbolic conditions on the final value g in $\mathcal{H}$ and the weak parabolic one.

4. **Uniform Extensions to the interval** $[0,T]$

The study in the section above concerning the close relationship between convergence rates of the final data $g \in \mathcal{H}$, stability and error-estimates given at the time $t = T$, enables us to give in the following section the global framework respectively according to the uniform variation of the variable t in $[0,T]$. Consequences, conclusions, and comparisons with the recent literature are highlighted.

THEOREM 4.1. *Assume $g \in \mathcal{H}$ and $f \in L^2((0,T),H)$, the non-homogeneous (FVP) approximate scheme admits a solution $u(t)$ if and only if the sequence $u_\alpha(0)$ converges in $\mathcal{H}$. Moreover, $u_\alpha(t)$ approaches to $u(t)$ uniformly in t.*

P r o o f. Assume $\lim\limits_{\alpha \longrightarrow 0} u_\alpha(0)$ exists and is an element of $\mathcal{H}$. write

$$u(t) = \sum_{k\geq 1}\left(u_{0k} + \int_0^t e^{\lambda_k \tau} f_k(\tau)\, d\tau\right) e^{-\lambda_k t}\varphi_k,$$

where

$$u_0 = \sum_{k\geq 1} u_{0k}\varphi_k.$$

Let $t \in [0,T]$, from (4) we have

$$\|u_\alpha(t) - u(t)\|^2 = \sum_{k\geq 1}(u_{\alpha k}(t) - u_k(t))^2$$

$$= \sum_{k\geq 1}\left[\frac{e^{\lambda_k(T-t)}}{\alpha e^{\lambda_k^P T}+1}\left(g_k - \int_t^T e^{\lambda_k(\tau-T)} f_k(\tau)\, d\tau\right) - u_{0k}e^{-\lambda_k t}\right.$$

$$\left. -u_{0k}e^{-\lambda_k t} - e^{-\lambda_k t}\int_0^t e^{\lambda_k \tau} f_k(\tau)\, d\tau\right]^2.$$

Assembling the terms, we have

$$\|u_\alpha(t)-u(t)\|^2=\sum_{k\geq 1}\left[e^{-\lambda_k t}\left[\frac{e^{\lambda_k T}}{\alpha e^{\lambda_k^P T}+1}\left(g_k-\int_t^T e^{\lambda_k(\tau-T)}f_k(\tau)\,d\tau\right)\right.\right.$$
$$\left.\left.-u_{0k}\right]-\frac{\alpha e^{\lambda_k T}}{\alpha e^{\lambda_k^P T}+1}\int_0^t e^{\lambda_k(\tau-t)}f_k(\tau)\,d\tau\right]^2$$

which implies that,

$$\|u_\alpha(t)-u(t)\|^2\leq 2\,\|u_\alpha(0)-u(0)\|^2$$
$$+\,2\sum_{k\geq 1}\left(\frac{\alpha e^{\lambda_k^p T}}{\alpha e^{\lambda_k^P T}+1}\right)^2\left(\int_0^t e^{\lambda_k(\tau-t)}f_k(\tau)\,d\tau\right)^2,$$

because of

$$\left(\int_0^t e^{\lambda_k(\tau-t)}f_k(\tau)\,d\tau\right)^2\leq\int_0^t e^{2\lambda_k(\tau-t)}d\tau\int_0^t f_k^2(\tau)\,d\tau$$
$$\leq\frac{1}{2\lambda_k}\|f_k\|^2_{L^2(0,T)},$$

we get

$$\sum_{k\geq 1}\left(\frac{e^{\lambda_k T}}{\alpha e^{\lambda_k^P T}+1}\right)^2\left(\int_0^t e^{\lambda_k(\tau-t)}f_k(\tau)\,d\tau\right)^2$$
$$\leq\frac{1}{2}\|f_k\|^2_{L^2(0,T)}\sum_{k\geq 1}\left(\frac{\alpha e^{\lambda_k^p T}}{\sqrt{\lambda_k}\left(\alpha e^{\lambda_k^P T}+1\right)}\right)^2,$$

since

$$\frac{\alpha e^{\lambda_k T}}{\sqrt{\lambda_k}\left(\alpha e^{\lambda_k^P T}+1\right)}\leq\frac{\alpha e^{\lambda_k T}}{\alpha\sqrt{\lambda_k}e^{\lambda_k^P T}+1},\ \lambda>1,$$

from the supremum estimates on $h_4(\lambda)$ in Theorem 3.6, with $s=\frac{1}{2}$ the function :

$$h_5(\lambda)=\frac{\alpha e^{\lambda_k T}}{\alpha\sqrt{\lambda_k}e^{\lambda_k^P T}+1},$$

satisfies,

$$\sup_{\lambda\geq 0}\alpha h_5(\lambda)\leq\frac{1}{\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{2p}}}.$$

Summing up these results, we deduce that,

$$\|u_\alpha(t) - u(t)\|^2 \leq 2\|u_\alpha(0) - u(0)\|^2$$

$$+\frac{1}{\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}\|f\|^2_{L^2((0,T),\mathcal{H})},$$

thus $u_\alpha(t)$ converges uniformly to $u(t)$ in $\mathcal{H}$. Particularly if $t = T$, then $\lim\limits_{\alpha\longrightarrow 0} u_\alpha(T) = u(T)$, using Theorem 3.3 we have well that $u(T) = g$. Conversely, if the non-homogeneous (FVP)-problem admits a solution $u(t)$, then from Lemma 2.1, we have

$$\sum_{k\geq 1}\left(g_k - \int_0^T e^{\lambda_k(\tau - T)} f_k(\tau)\, d\tau\right)^2 e^{2\lambda_k T} < +\infty.$$

Then we write

$$\|u_\alpha(0) - u_\gamma(0)\|^2 \leq \frac{(\alpha-\gamma)^2}{(\alpha+\gamma)^2}\sum_{k\geq 1} e^{2\lambda_k T}\left(g_k - \int_0^T e^{\lambda_k(\tau-T)} f_k(\tau)\, d\tau\right)^2,$$

therefore $\sum\limits_{k\geq 1} e^{2\lambda_k T}\left(g_k - \int_0^T e^{\lambda_k(\tau-T)} f_k(\tau)\, d\tau\right)^2 < +\infty$, implies that $\{u_\alpha(0)\}$ is a Cauchy sequence, hence convergent in $\mathcal{H}$, which leads to required statement. □

THEOREM 4.2. *Assume that $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$. Suppose the non-homogeneous (FVP) regular approximating process admits a classical solution $u(t)$ which verifies $\left\|e^{rTA^\epsilon}u(t)\right\| < +\infty$, for arbitrary parameters r and ϵ satisfying $r > 0$ and $0 < \epsilon \leq p$, then*

$$\|u_\alpha(t) - u(t)\| \leq c e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{\epsilon}{p}}}, \tag{16}$$

for each $t \in [0,T]$, where c is a constant depending on $\sup\limits_{t\in[0,T]}\left\|e^{rTA^\epsilon}u(t)\right\|_{\mathcal{H}}$.

P r o o f. Let the (FVP) approximating scheme admit a unique classical solution, from the expressions (3),(5) of $u(t)$ and $u_\alpha(t)$ we have

$$\begin{aligned} &\|u_\alpha(t) - u(t)\|^2 \\ &= \sum_{k\geq 1} \frac{\alpha^2 e^{2\lambda_k^p T}}{\left(\alpha e^{\lambda_k^p T}+1\right)^2} \left(g_k e^{\lambda_k(T-t)} - \int_t^T e^{\lambda_k(\tau-t)} f_k(\tau)\, d\tau\right)^2 \\ &= \sum_{k\geq 1} \left(\frac{\alpha e^{\left(\lambda_k^p - r\lambda_k^\epsilon\right)T}}{\alpha e^{\lambda_k^p T}+1}\right)^2 e^{2r\lambda_k^\epsilon T} \left(g_k e^{\lambda_k(T-t)} - \int_t^T e^{\lambda_k(\tau-t)} f_k(\tau)\, d\tau\right)^2, \end{aligned}$$

From the supremum inequality (12) in Theorem 3.4, we get

$$\|u_\alpha(t) - u(t)\| \leq c e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{\epsilon}{p}}}, \ \forall t \in [0,T],$$

where the constant c depends on $\sup\limits_{t\in[0,T]} \left\|e^{rTA^\epsilon} u(t)\right\|_{\mathcal{H}}$, as required. □

REMARK 4.1. For the parametric values of $r > 0$, and $0 < \epsilon < p$ the order of convergence is $e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{\epsilon}{p}}}$ which satisfies,

$$e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{\epsilon}{p}}} < \left[\sum_{n=0}^{m} c_n(r,T) \left(\frac{1}{\log\frac{1}{\alpha}}\right)^{\frac{n\epsilon}{p}}\right]^{-1}, \ \forall m \in \mathbb{N}^*, \tag{17}$$

it is new in the literature, it is finer than any order of polynomial powers of known logarithmic types.

THEOREM 4.3. *Let the final data $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$, assume that $\left\|e^{rAT}u(t)\right\| < +\infty$, $r > 0$, then*

$$\|u_\alpha(t) - u(t)\| \leq c e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}, \ \forall t \in [0,T], \tag{18}$$

where c depends on $\sup\limits_{t\in[0,T]} \left\|e^{rTA} u(t)\right\|_{\mathcal{H}}$.

P r o o f. Following the same steps as those of Theorem 4.2, we have:

$$\begin{aligned} \|u_\alpha(t) - u(t)\|^2 = \sum_{k\geq 1} &\left(\frac{\alpha e^{\left(\lambda_k^p - r\lambda_k\right)T}}{\alpha e^{\lambda_k^p T}+1}\right)^2 e^{2r\lambda_k T} \\ &\times \left(g_k e^{\lambda_k(T-t)} - \int_t^T e^{\lambda_k(\tau-t)} f_k(\tau)\, d\tau\right)^2, \end{aligned}$$

using the supremum inequality (13) in Theorem 3.5, we get

$$\|u_\alpha(t) - u(t)\|^2 \leq c^2 e^{-2rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}},$$

which is the desired result, the constant c depending on $\sup_{t\in[0,T]} \left\|e^{rTA}u(t)\right\|_{\mathcal{H}}$. □

REMARK 4.2. Under a weaker hyperbolic condition, this theorem also verifies

$$e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}} < \left[\sum_{n=0}^{m} c_n(r,T)\left(\frac{1}{\log\frac{1}{\alpha}}\right)^{\frac{n}{p}}\right]^{-1},\ \forall m \in \mathbb{N}^*, \tag{19}$$

which is faster than any polynomial powers of logarithmic types.

Next, for weaker parabolic conditions, we have the results below.

THEOREM 4.4. *Let $g \in \mathcal{H}$ and $f \in L^2((0,T), \mathcal{H})$, assume that $\|A^s u(t)\| < +\infty$, $s > 0$, then*

$$\|u_\alpha(t) - u(t)\| \leq c\frac{1}{\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{s}{p}}},\ \forall t \in [0,T], \tag{20}$$

where the constant c depends on $\sup_{t\in[0,T]} \|A^s u(t)\|_{\mathcal{H}}$.

P r o o f. As in the preceding theorems, we write

$$\begin{aligned}
&\|u_\alpha(t) - u(t)\|^2 \\
&= \sum_{k\geq 1}\left(\frac{\alpha e^{\lambda_k^p T}}{\alpha e^{\lambda_k^p T}+1}\right)^2\left(g_k e^{\lambda_k(T-t)} - \int_t^T e^{\lambda_k(\tau-t)} f_k(\tau)\,d\tau\right)^2 \\
&= \sum_{k\geq 1}\left(\frac{\alpha e^{\lambda_k^p T}}{\lambda_k^s\left(\alpha e^{\lambda_k^p T}+1\right)}\right)^2 \lambda_k^{2s}\left(g_k e^{\lambda_k(T-t)} - \int_t^T e^{\lambda_k(\tau-t)} f_k(\tau)\,d\tau\right)^2,
\end{aligned}$$

then using the supremum estimate (15) in Theorem 3.6, we deduce

$$\|u_\alpha(t) - u(t)\|^2 \leq c^2\frac{1}{\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{2s}{p}}},\ \forall t \in [0,T],$$

which is the desired result, the constant c depends on $\sup_{t\in[0,T]} \|A^s u(t)\|_{\mathcal{H}}$.
□

Now, according to the previous Theorems 4.2, 4.3 and 4.4, we may establish three kinds of convergence results for given non-exact data, corresponding respectively to the hyperbolic or parabolic conditions on the solution $u(t)$. In this direction, this depends upon the various needs in applications. For instance, let us set the following;

THEOREM 4.5. *Let $g \in \mathcal{H}$ and $f \in L^2((0,T),\mathcal{H})$, assume the (FVP)-problem admits a unique solution $u(t)$ with $\left\|e^{rAT}u(t)\right\| < +\infty$. Let g_α be a measured data satisfying $\|g - g_\alpha\| \leq \alpha$,then there exists $v_\alpha(t)$ solution associated to this final value satisfying*

$$\|v_\alpha(t) - u(t)\| \leq \alpha e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}} + ce^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}, \tag{21}$$

that is for each $t \in [0,T]$, where c is a constant depending upon $\sup\limits_{t\in[0,T]}\left\|e^{rTA}u(t)\right\|_{\mathcal{H}}$.

P r o o f. If $u_\alpha(t)$ and $v_\alpha(t)$ are solutions of the approximate (AFVP)-process associated respectively to g and g_α, then

$$\|v_\alpha(t) - u(t)\| \leq \|v_\alpha(t) - u_\alpha(t)\| + \|u_\alpha(t) - u(t)\|,$$

using the stability in Theorem 3.1 and the error-estimate in Theorem 4.3, we obtain

$$\|v_\alpha(t) - u(t)\| \leq \alpha e^{T\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}} + ce^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}},$$

where c depends upon $\sup\limits_{t\in[0,T]}\left\|e^{rTA}u(t)\right\|_{\mathcal{H}}$. After few simplifications, this order is of type

$$\|v_\alpha(t) - u(t)\| \leq c_1 e^{-rT\left(\frac{1}{T}\log\frac{1}{\alpha}\right)^{\frac{1}{p}}}.$$

□

The regularization proposed in this work yields stability and convergence estimates significantly better than classical Hölder and logarithmic-type results. As emphasized in Remark 4.2, the obtained order is better than any polynomial power of logarithmic type, and the regularized problem is more stable than the Hölder one $\alpha^{-\frac{1}{s}}$, $s \geq 1$.

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