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NONLINEAR SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL EQUATIONS WITH A ZERO OPERATOR OF THE DIFFERENTIAL PART AND EXPONENTIALLY OSCILLATING INHOMOGENEITY

Abdukhafiz Bobodzhanov [1], **Burkhan Kalimbetov** [2,§], **Valeriy Safonov** [1], **Dinmukhambet Sapakov** [2]

[1] National Research University, MPEI

Krasnokazarmennaya 14, Moscow, 111250, RUSSIA

e-mails: bobojanova@mpei.ru, Singsaf@yandex.ru

[2] Kuatbekov Peoples' Friendship University

Tole bi 32b, Shymkent, 160011, KAZAKHSTAN

e-mails: bkalimbetov@mail.ru ([§] corresponding author),

sapakov1986@mail.ru

Abstract

In this paper, we consider a nonlinear integro-differential problem with a zero operator of the differential part, the integral operator of which contains a rapidly changing kernel, and the right-hand side depends on a rapidly oscillating exponent. This work is a continuation of the research carried out earlier for a similar linear system with a rapidly changing kernel. In the nonlinear case, the conditions for the solvability of the

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corresponding iterative problems, as in the linear case, will have the form not of differential (as was the case in problems with a nonzero operator of the differential part), but of integro-differential equations, and the formation of these equations is played by nonlinearity and rapidly oscillating inhomogeneity.

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1. **Introduction**

In this work, the regularization method of S.A. Lomov [28, 29] is generalized for problems for an integro-differential equation with a rapidly changing kernel and with a right-hand side depending on a rapidly oscillating exponent

$$\begin{aligned} &\varepsilon \tfrac{dy}{dt} = \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds + \varepsilon f\,(y,t) \\ &+h_1(t) + h_2(t)e^{\frac{i\beta(t)}{\varepsilon}}, \quad y(0,\varepsilon) = y^0,\, t \in [0,T]. \end{aligned} \tag{1.1}$$

The work is a continuation of the research carried out earlier for a linear problem with a zero operator of the differential part [4], as well as for nonlinear integro-differential equations with a zero operator of the differential part and with several rapidly changing kernels [5]. In the case of the problem (1.1), the singularities in its solution are described by the spectral values of the kernel and rapidly oscillating inhomogeneities. However, the influence of the zero operator of the differential part affects the fact that in the first approximation the asymptotics of the solution of the problem under consideration will not contain boundary layer functions, and the limit operator itself will be degenerate (but not zero). In this case, the conditions for the solvability of the corresponding iterative problems, as in the linear case, will have the form not of differential (as was the case in problems with a nonzero operator of the differential part), but of integro-differential equations, and nonlinearity plays an essential role in the formation of these equations. In this case, so-called resonances can arise, which significantly complicate the development of the corresponding algorithm of the regularization method. In this paper, we consider the nonresonant case. It is assumed that the study of an alternative variant (a more complex resonance problem) will be carried out in the future.

Problem (1.1) is considered under the following conditions:

(i) $\mu(t), \beta'(t) \in \mathrm{C}^{\infty}([0,T], \mathbf{R})$, $\mathbf{h_1(t)}, \mathbf{h_2(t)} \in \mathrm{C}^{\infty}\left([0,\mathrm{T}], \mathbf{C}\right)$, $K(t,s) \in \mathrm{C}^{\infty}(0 \leq s \leq t \leq T, \mathbf{R})$;

(ii) $\mu(t) < 0, \beta'(t) > 0 \ \ \forall t \in [0,T]$;

(iii) $f(y,t)$ is a polynomial, i.e.

$$f(y,t) = \sum_{m=0}^{N} f_m(t) y^m$$

with the coefficients $f_m(t) \in C^{\infty}\left([0,T], \mathbf{R}\right), m = \overline{0,N}, N < \infty$.

As mentioned earlier, this work is a continuation of the studies [6, 7] carried out earlier for one rapidly changing kernel. In contrast to the linear case, on the right-hand side of problem (1.1), there is no inhomogeneity $g(t)$ (without a coefficient ε). Its presence in the problem (1.1) would entail the appearance in the asymptotic solution of terms with negative powers of the parameter ε, and in the nonlinear case there would be an innumerable set of such powers, and the corresponding formal asymptotic solution would have the form of a Laurent series. This would make the development of an algorithm for asymptotic solutions problematic; therefore, in the present work, wishing to remain within the framework of asymptotic solutions of the Taylor series type, inhomogeneity $g(t)$ (without a coefficient ε) is excluded.

A generalization of the idea of the regularization method for singularly perturbed integral and integro-differential equations with rapidly oscillating coefficients is considered in [8, 9, 13, 17, 19, 20, 21], with slowly varying coefficients, but with rapidly oscillating inhomogeneities, in [3, 10, 11, 12, 14, 15, 16, 18, 25, 27].Singularly perturbed integro-differential partial differential equations have been studied in [22, 23, 26], and singularly perturbed differential, integro-differential equations with fractional derivatives - in papers [1, 24].

Turning to the development of the algorithm, we note that throughout the work, column vectors are written in curly brackets, and row vectors are written in simple brackets.

2. Equivalent integro-differential system and its regularization

We introduce a new unknown function

$$z = \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds.$$

Differentiating it by t, we will have

$$\begin{array}{l} \frac{dz}{dt} = K(t,t)y + \frac{\mu(t)}{\varepsilon}\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds \\ +\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} \frac{\partial K(t,s)}{\partial t} y(s,\varepsilon)ds \\ \Leftrightarrow \varepsilon\frac{dz}{dt} = \mu(t)z + \varepsilon K(t,t)y + \varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} \frac{\partial K(t,s)}{\partial t} y(s,\varepsilon)ds. \end{array}$$

Instead of (1.1), we get the system

$$\begin{array}{l} \varepsilon\frac{dw}{dt} = A(t)w + \varepsilon A_1(t)w + \varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} G(t,s)w(s,\varepsilon)ds \\ +\varepsilon F(w,t) + H(t,\varepsilon), \quad w(0,\varepsilon) = w^0 \equiv \{y^0, 0\}, \end{array} \tag{2.1}$$

where $w = \{y,z\}$, $F(w,t) = \{f(y,t), 0\}$, $H(t,\varepsilon) = \{h_1(t) + h_2(t)e^{\frac{i\beta(t)}{\varepsilon}}, 0\}$, and the matrices $A(t), A_1(t), G(t,s)$ are of the form

$$\begin{array}{l} A(t) = \begin{pmatrix} 0 & 1 \\ 0 & \mu(t) \end{pmatrix}, A_1(t) = \begin{pmatrix} 0 & 0 \\ K(t,t) & 0 \end{pmatrix}, \\ G(t,s) = \begin{pmatrix} 0 & 0 \\ \frac{\partial K(t,s)}{\partial t} & 0 \end{pmatrix}. \end{array}$$

For convenience, let us denote $\lambda_1(t) \equiv i\beta'(t)$, $\lambda_2(t) \equiv \mu(t)$. We introduce regularizing variables (see [28])

$$\tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, \quad j = 1, 2,$$

and consider the following problem:

$$
\begin{aligned}
&\varepsilon\tfrac{\partial\tilde{w}}{\partial t}+\sum_{j=1}^{2}\lambda_j(t)\tfrac{\partial\tilde{w}}{\partial\tau_j}-A(t)\tilde{w}-\varepsilon A_1(t)\tilde{w}-\varepsilon F(\tilde{w},t)H(t,\tau)\\
&=\varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_2(\theta)d\theta}G(t,s)\tilde{w}(s,\tfrac{\psi(s)}{\varepsilon},\varepsilon)ds,\\
&\tilde{w}(t,\tau\ ,\sigma,\varepsilon)\big|_{t=0,\tau=0,\sigma=e^{\frac{i\beta(0)}{\varepsilon}}}=w^0,
\end{aligned}
\tag{2.2}
$$

where $H(t,\tau)=\{h_1(t)+h_2(t)e^{\tau_1}\sigma,0\}$, $\sigma=e^{\frac{i}{\varepsilon}\beta(0)}$, for the function $\tilde{w}=\tilde{w}(t,\tau,\sigma,\varepsilon)$, where $\tau=(\tau_1,\tau_2)$, $\psi=(\psi_1,\psi_2)$. It is clear that if $\tilde{w}=\tilde{w}(t,\tau,\sigma,\varepsilon)$ is the solution to problem (2.2), then the vector function $w=\tilde{w}\left(t,\frac{\psi(t)}{\varepsilon},\sigma,\varepsilon\right)$ is an exact solution to problem (2.1); therefore, the problem (2.2) is extended with respect to the problem (2.1).

However, the problem (2.2) cannot be considered completely regularized, since the integral operator

$$
\begin{aligned}
J\tilde{w}&\equiv J\left(\tilde{w}\left(t,\tau,\varepsilon\right)\big|_{t=s,\tau=\frac{\psi(s)}{\varepsilon}}\right)\\
&=\varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_2(\theta)d\theta}G(t,s)\tilde{w}\left(s,\frac{\psi(s)}{\varepsilon},\varepsilon\right)ds
\end{aligned}
$$

is not regularized. To regularize it, we introduce a class $M_\varepsilon==U|_{\tau=\frac{\psi(t)}{\varepsilon}}$, that is asymptotically invariant with respect to the operator J (see [28], p. 62]).

In this case, we take U as the space of vector functions representable by sums of the form

$$
\begin{aligned}
&w(t,\tau,\sigma)=w_0(t,\sigma)+\sum_{j=1}^{2}w_j(t,\sigma)e^{\tau_j}\\
&+\sum_{|m|\geq 2}w^{(m)}(t,\sigma)e^{(m,\tau)},w_j(t,\sigma),w^{(m)}(t,\sigma)\\
&\in C^\infty\left([0,T],\mathbf{R}\right),\quad j=\overline{0,2},\ \ 2\leq|m|\leq N_w.
\end{aligned}
\tag{2.3}
$$

Here $m=(m_1,m_2)$ is a multi-index, $|m|=m_1+m_2$. Let us show that the class M_ε, is asymptotically invariant with respect to the operator J. To do this, it is necessary to show that the image $Jw(t,\tau)$ on functions of the form (2.3) can be represented as a series

$$
Jw(t,\tau)=\sum_{k=0}^{\infty}\varepsilon^k\left(\sum_{j=1}^{2}w_k(t)e^{\tau_j}+w_k^{(0)}(t)\right)\Bigg|_{\tau=\frac{\psi(t)}{\varepsilon}}
$$

converging asymptotically to Jw (as $\varepsilon \to +0$) uniformly with respect to $t \in [0, T]$. Substituting (2.3) in $Jw(t, \tau)$, we have

$$Jw(t,\tau) = e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t G(t,s)w_0(s)e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_2(\theta)d\theta} ds$$

$$+e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t G(t,s)w_1(s)e^{\frac{1}{\varepsilon}\int\limits_0^s (\lambda_1(\theta)-\lambda_2(\theta))d\theta} ds$$

$$+e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t G(t,s)w_2(s)ds$$

$$+\sum_{|m|\geq 2} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t G(t,s)w^{(m)}(s)e^{\frac{1}{\varepsilon}\int\limits_0^s [(m,\lambda(\theta))-\lambda_2(\theta)]d\theta} ds.$$

By applying integration by parts, we have

$$J_0(t,\varepsilon) = \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t \frac{G(t,s)w_0(s)}{-\lambda_2(s)} de^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_2(\theta)d\theta}$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \left[\left. \frac{G(t,s)w_0(s)}{-\lambda_2(s)}\, e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_2(\theta)d\theta} \right|_{s=0}^{s=t} \right.$$

$$\left. -\int\limits_0^t \left(\frac{\partial}{\partial s}\frac{G(t,s)w_0(s)}{-\lambda_2(s)}\right) e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_2(\theta)d\theta} ds \right]$$

$$= \varepsilon \left[\frac{G(t,t)w_0(t)}{-\lambda_2(t)} - \frac{G(t,0)w_0(0)}{-\lambda_2(0)} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \right]$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t \left(\frac{\partial}{\partial s}\frac{G(t,s)w_0(s)}{-\lambda_2(s)}\right) e^{-\frac{1}{\varepsilon}\int\limits_0^s \lambda_2(\theta)d\theta} ds.$$

Continuing this process further, we will have

$$J_0(t,\varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon^{\nu+1}(-1)^{\nu} \left[\left(I_0^{\nu}(G(t,s)w_0(s))\right)_{s=t} \right.$$

$$\left. -\left(I_0^{\nu}(G(t,s)w_0(s))\right)_{s=0} e^{\frac{1}{\varepsilon}\int\limits_0^s \lambda_2(\theta)d\theta} \right],$$

where

$$I_0^0 = \frac{1}{-\lambda_2(s)},\ I_0^{\nu} = \frac{1}{-\lambda_2(s)}\frac{\partial}{\partial s} I_0^{\nu-1} \quad (\nu \geq 1);$$

$$J_1(t,\varepsilon) = \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t \frac{G(t,s)w_1(s)}{\lambda_1(s)-\lambda_2(s)} de^{\frac{1}{\varepsilon}\int\limits_0^s (\lambda_1(\theta)-\lambda_2(\theta))d\theta}$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \left[\frac{G(t,s)w_1(s)}{\lambda_1(s)-\lambda_2(s)} \, e^{\frac{1}{\varepsilon}\int\limits_0^s (\lambda_1(\theta)-\lambda_2(\theta))d\theta} \Bigg|_{s=0}^{s=t} \right.$$

$$\left. - \int\limits_0^t \left(\frac{\partial}{\partial s} \frac{G(t,s)w_1(s)}{\lambda_1(s)-\lambda_2(s)} \right) e^{\frac{1}{\varepsilon}\int\limits_0^s (\lambda_1(\theta)-\lambda_2(\theta))d\theta} ds \right]$$

$$= \varepsilon \left[\frac{G(t,t)w_1(t)}{\lambda_1(t)-\lambda_2(t)} - \frac{G(t,0)w_1(0)}{\lambda_1(0)-\lambda_2(0)} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \right]$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t \left(\frac{\partial}{\partial s} \frac{G(t,s)w_1(s)}{\lambda_1(s)-\lambda_2(s)} \right) e^{\frac{1}{\varepsilon}\int\limits_0^s (\lambda_1(\theta)-\lambda_2(\theta))d\theta} ds.$$

Continuing this process further, we will have

$$J_1(t,\varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon^{\nu+1}(-1)^{\nu} \left[(I_1^{\nu}(G(t,s)w_1(s)))_{s=t} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_1(\theta)d\theta} \right.$$

$$\left. -(I_1^{\nu}(G(t,s)w_1(s)))_{s=0} e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \right],$$

where

$$I_1^0 = \frac{1}{\lambda_1(s)-\lambda_2(s)}, \;\; I_1^{\nu} = \frac{1}{\lambda_1(s)-\lambda_2(s)} \frac{\partial}{\partial s} I_1^{\nu-1} \quad (\nu \geq 1).$$

And similarly, we will have:

$$J_m(t,\varepsilon) = \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \int\limits_0^t \frac{G(t,s)w^{(m)}(s)}{(m,\lambda(s))-\lambda_2(s)} de^{\frac{1}{\varepsilon}\int\limits_0^s ((m,\lambda(\theta))-\lambda_2(\theta))d\theta}$$

$$= \varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_2(\theta)d\theta} \left[\frac{G(t,s)w^{(m)}(s)}{(m,\lambda(s))-\lambda_2(s)} \, e^{\frac{1}{\varepsilon}\int\limits_0^s ((m,\lambda(\theta))-\lambda_2(\theta))d\theta} \Bigg|_{s=0}^{s=t} \right.$$

$$\left. - \int\limits_0^t \left(\frac{\partial}{\partial s} \frac{G(t,s)w^{(m)}(s)}{(m,\lambda(s))-\lambda_2(s)} \right) e^{\frac{1}{\varepsilon}\int\limits_0^s ((m,\lambda(\theta))-\lambda_2(\theta))d\theta} ds \right]$$

$$= \varepsilon\left[\frac{G(t,t)w^{(m)}(t)}{(m,\lambda(t))-\lambda_2(t)}e^{\frac{1}{\varepsilon}\int\limits_0^t(m,\lambda(\theta))d\theta} - \frac{G(t,0)w^{(m)}(0)}{(m,\lambda(0))-\lambda_2(0)}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_2(\theta)d\theta}\right]$$

$$-\varepsilon e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_2(\theta)d\theta}\int\limits_0^t\left(\frac{\partial}{\partial s}\frac{G(t,s)w^{(m)}(s)}{(m,\lambda(s))-\lambda_2(s)}\right)e^{\frac{1}{\varepsilon}\int\limits_0^s((m,\lambda(\theta))-\lambda_2(\theta))d\theta}ds.$$

Continuing this process further, we will have

$$J_m(t,\varepsilon)=\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}(-1)^{\nu}\left[\left(I_m^{\nu}(G(t,s)w^{(m)}(s))\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_0^t(m,\lambda(\theta))d\theta}\right.$$
$$\left.-\left(I_m^{\nu}(G(t,s)w^{(m)}(s))\right)_{s=0}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_2(\theta)d\theta}\right],$$

where

$$I_m^0=\frac{1}{(m,\lambda(s))-\lambda_2(s)},\ I_m^{\nu}=\frac{1}{(m,\lambda(s))-\lambda_2(s)}\frac{\partial}{\partial s}I_m^{\nu-1},$$
$$2\le|m|\le N_w\quad(\nu\ge1).$$

Hence, the image of the operator J on an element (2.3) of the space U can be represented as a series

$$Jw(t,\tau)=e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_2(\theta)d\theta}\int\limits_0^t G(t,s)w_2(s)ds$$
$$+\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}(-1)^{\nu}\left[(I_0^{\nu}(G(t,s)w_0(s)))_{s=t}\right.$$
$$\left.-(I_0^{\nu}(G(t,s)w_0(s)))_{s=0}e^{\frac{1}{\varepsilon}\int\limits_0^s\lambda_2(\theta)d\theta}\right]$$
$$+\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}(-1)^{\nu}\left[(I_1^{\nu}(G(t,s)w_1(s)))_{s=t}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_1(\theta)d\theta}\right.$$
$$\left.-(I_1^{\nu}(G(t,s)w_1(s)))_{s=0}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_2(\theta)d\theta}\right]$$
$$+\sum_{|m|\ge2}^{N_w}\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}(-1)^{\nu}\left[\left(I_m^{\nu}(G(t,s)w^{(m)}(s))\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_0^t(m,\lambda(\theta))d\theta}\right.$$
$$\left.-\left(I_m^{\nu}(G(t,s)w^{(m)}(s))\right)_{s=0}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_2(\theta)d\theta}\right].$$

It is easy to show (see, for example, [30], pp. 291-294) that this series converges asymptotically for $\varepsilon \to +0$ (uniformly in $t \in \in [0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon \to +0$) with respect to the operator J.

Let us introduce the operators $R_\nu : U \to U$, acting on each element $y(t,\tau) \in U$ of the form (2.3) according to the law:

$$\begin{aligned}
&R_0 w(t,\tau) = e^{\tau_2} \int\limits_0^t G(t,s) w_2(s) ds,\\
&R_1 w(t,\tau) = \left[\left(I_0^0\left(G(t,s)w_0(s)\right)\right)_{s=t} - \left(I_0^0\left(G(t,s)w_0(s)\right)\right)_{s=0} e^{\tau_2}\right]\\
&+\left[\left(I_1^0\left(G(t,s)w_1(s)\right)\right)_{s=t} e^{\tau_1} - \left(I_1^0\left(G(t,s)w_1(s)\right)\right)_{s=0} e^{\tau_2}\right]\\
&+\sum_{|m|\geq 2}^{N_w} \left[\left(I_m^0 (G(t,s)w^{(m)}(s))\right)_{s=t} e^{(m,\tau)} - \left(I_m^0 (G(t,s)w^{(m)}(s))\right)_{s=0} e^{\tau_2}\right],
\end{aligned}$$

$$\begin{aligned}
&R_{\nu+1} w(t,\tau) = (-1)^\nu \left[\left(I_0^\nu\left(G(t,s)w_0(s)\right)\right)_{s=t} - \left(I_0^\nu\left(G(t,s)w_0(s)\right)\right)_{s=0} e^{\tau_2}\right]\\
&\quad +(-1)^\nu \left[\left(I_1^\nu\left(G(t,s)w_1(s)\right)\right)_{s=t} e^{\tau_1} - \left(I_1^\nu\left(G(t,s)w_1(s)\right)\right)_{s=0} e^{\tau_2}\right]\\
&\quad +\sum_{|m|\geq 2}^{N_w} (-1)^\nu \left[\left(I_m^\nu (G(t,s)w^{(m)}(s))\right)_{s=t} e^{(m,\tau)}\right. \qquad (2.4)\\
&\quad \left. - \left(I_m^\nu (G(t,s)w^{(m)}(s))\right)_{s=0} e^{\tau_2}\right], \quad \nu \geq 1.
\end{aligned}$$

Then the image $Jw(t,\tau)$ can be written in the form

$$Jw(t,\tau) = R_0 w(t,\tau) + \sum_{m=0}^{\infty} \varepsilon^{m+1} R_{m+1} w(t,\tau), \tag{2.5}$$

where $\tau = \frac{\psi(t)}{\varepsilon}$. Let us now extend the operator J to series of the form

$$\tilde{w}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} w_k(t,\tau) \tag{2.6}$$

with coefficients $w_k(t,\tau) \in U, k \geq 0$.

A formal extension $\tilde{J}$ of the operator J on series of the form (2.6) is the operator

$$\tilde{J}\tilde{w}(t,\tau,\varepsilon) \overset{def}{=} \sum_{\nu=0}^{\infty} \varepsilon^\nu \sum_{s=0}^{\nu} R_{\nu-s} w_s(t,\tau). \tag{2.7}$$

Despite the fact that the extension $\tilde{J}$ of the operator J is defined formally, it is quite possible to use it in constructing an asymptotic solution of finite order of ε. Now it is easy to write out the regularized (with respect to

(2.2)) problem:

$$\begin{aligned}&\varepsilon\tfrac{\partial\tilde{w}}{\partial t}+\sum_{j=1}^{2}\lambda_j(t)\tfrac{\partial\tilde{w}}{\partial\tau_j}-A(t)\tilde{w}-\varepsilon A_1(t)\tilde{w}-\varepsilon\tilde{J}\tilde{w}\\&=\varepsilon F(\tilde{w},t)+H(t,\tau),\quad \tilde{w}(t,\tau\ ,\sigma,\varepsilon)\big|_{t=0,\tau=0,\sigma=e^{\frac{i\beta(0)}{\varepsilon}}}=w^0,\end{aligned}\tag{2.8}$$

where $\tilde{J}$ is the operator (2.7).

3. Solvability of iterative problems

Substituting series (2.7) into (2.8) and equating the coefficients at the same degrees of ε, we obtain the following iterative problems:

$$\begin{aligned}L_0w_0(t,\tau)&\equiv\sum_{j=1}^{2}\lambda_j(t)\tfrac{\partial w_0}{\partial\tau_j}-A(t)w_0\\&=\begin{pmatrix}h_1(t)+h_2(t)e^{\tau_1}\sigma\\0\end{pmatrix},\quad w_0(0,0)=w^0;\end{aligned}\tag{3.1_0}$$

$$\begin{aligned}L_0w_1(t,\tau)=-\frac{\partial w_0}{\partial t}+A_1(t)w_0+F(w_0,t)+R_0w_0,\\ w_1(0,0)=0;\end{aligned}\tag{3.1_1}$$

$$\begin{aligned}L_0w_2(t,\tau)=-\frac{\partial w_1}{\partial t}+A_1(t)w_1+\frac{\partial F(w_0,t)}{\partial t}w_1+R_0w_1\\ +R_1w_0,\, w_2(0,0)=0;\end{aligned}\tag{3.1_2}$$

$$\cdots\quad\cdots\quad\cdots$$

$$\begin{aligned}&L_0w_k(t,\tau)=-\tfrac{\partial w_{k-1}}{\partial t}+A_1(t)w_{k-1}+P_k(w_0,\dots,w_{k-1},t)\\&+R_0w_{k-1}+R_1w_{k-2}+\ldots+R_kw_0,\ \ w_k(0,0)=0,\ k\geq1,\end{aligned}\tag{3.1_k}$$

where $P_k(w_0,...,w_{k-1},t)$ is some polynomial of $w_0,\dots,w_{k-1}$, linear with respect to w_{k-1}.

Passing to the formulation of theorems on the normal and unique solvability of the iterative problems (3.1_k), we calculate the eigenvectors $\varphi_j(t)$ and $\chi_j(t)$ of matrices $A(t)$ and $A^*(t)$, respectively. It is easy to check that they have the form

$$\begin{aligned}&\varphi_0(t)=\{1,0\},\quad\varphi_2(t)=\left\{\tfrac{1}{\mu(t)},1\right\}\equiv\left\{\tfrac{1}{\lambda_2(t)},1\right\},\\&\chi_0(t)=\left\{1,-\tfrac{1}{\mu(t)}\right\}\equiv\left\{1,-\tfrac{1}{\lambda_2(t)}\right\},\ \ \chi_2(t)=\{0,1\},\end{aligned}\tag{3.2}$$

and $\varphi_0(t),\varphi_2(t)$ correspond to the eigenvalues $\lambda_0(t)\equiv0,\ \lambda_2(t)\equiv\equiv\mu(t)$ of the matrix $A(t)$, and $\chi_0(t),\chi_2(t)$ correspond to the eigenvalues $\bar{\lambda}_0(t)\equiv 0,\ \bar{\lambda}_2(t)\equiv\mu(t)$ of the matrix $A^*(t)$, respectively.

Each of the iterative systems has the form

$$L_0 w(t,\tau) \equiv \sum_{j=1}^{2} \lambda_j(t) \frac{\partial w}{\partial \tau_j} - A(t) w = P(t,\tau), \tag{3.3}$$

where $P(t,\tau) = P_0(t) + \sum\limits_{j=1}^{2} P_j(t) e^{\tau_j} + \sum\limits_{|m|\geq 2}^{N_P} P^{(m)}(t) e^{(m,\tau)} \in U.$

THEOREM 3.1. *Let conditions 1) – 3) be satisfied and* $P(t,\tau) \in \in U$. *For system (3.3) to have a solution in* U *it is necessary and sufficient that*

$$(\, P_j(t), \chi_j(t)) = 0, \;\; j = 0, 2, \;\; \forall t \in [0,T]. \tag{3.4}$$

P r o o f. We will define the solution of the system (3.3) in the form of the sum (2.4). Substituting (2.4) into (3.3), we obtain the equality

$$\begin{aligned} &\lambda_1(t) w_1(t) e^{\tau_1} + \lambda_2(t) w_2(t) e^{\tau_2} + \textstyle\sum_{|m|=2}^{N_w} m_1 w^{(m)}(t) e^{(m,\tau)} \\ &+ \sum_{|m|=2}^{N_w} m_2 w^{(m)}(t) e^{(m,\tau)} - A(t) w_0(t) - A(t) w_1(t) e^{\tau_1} - A(t) w_2(t) e^{\tau_2} \\ &- \sum_{|m|=2}^{N_w} A(t) w^{(m)}(t) e^{(m,\tau)} = P_0(t) + \sum_{j=1}^{2} P_j(t) e^{\tau_j} + \sum_{|m|\geq 2}^{N_P} P^{(m)}(t) e^{(m,\tau)}, \end{aligned}$$

or

$$\begin{aligned} &-A(t) w_0(t) + [\lambda_1(t) I - A(t)]\, w_1(t) e^{\tau_1} + [\lambda_2(t) I - A(t)]\, w_2(t) e^{\tau_2} \\ &+ \sum_{|m|=2}^{N_w} [(m,\lambda(t)) I - A(t)]\, w^{(m)}(t) e^{(m,\tau)} \\ &= \; P_0(t) + \textstyle\sum_{j=1}^{2} P_j(t) e^{\tau_j} + \sum_{|m|\geq 2}^{N_P} P^{(m)}(t) e^{(m,\tau)}. \end{aligned}$$

Due to the linear independence of the exponentials $e^{\tau_1}, \; e^{\tau_2}, \; e^{(m,\tau)}, \;\; (|m| \geq 2)$, this equality takes place only at $N_w = N_P$. Equating separately the coefficients at the same exponents and free terms, we will have

$$-A(t) w_0(t) = P_0(t), \tag{3.5_0}$$

$$[\lambda_j(t) I - A(t)]\, w_j(t) = P_j(t), \quad j = 1, 2, \tag{3.5_j}$$

$$[(m,\lambda(t)) I - A(t)]\, w^{(m)}(t) = P^{(m)}(t), \;\; 2 \leq |m| \leq N_P. \tag{3.5_m}$$

Let $\Phi(t) = (\varphi_0(t), \varphi_2(t))$ be the matrix of eigenvectors of the operator $A(t)$, and $\Lambda(t) = \mathrm{diag}(0, \lambda_2(t))$. In system (3.5_0) we make a transformation $w_0(t) = \Phi(t)\eta \equiv (\varphi_0(t), \varphi_2(t))\{\eta^1, \eta^2\}$, we obtain the system

$$-\Lambda(t)\eta = \Phi^{-1}(t)P_0(t) \Leftrightarrow \begin{cases} 0 \cdot \eta^1 = (P_0(t), \chi_0(t)), \\ -\lambda_2(t) \cdot \eta^2 = (P_0(t), \chi_2(t)). \end{cases}$$

For the solvability of this system, it is necessary and sufficient that identity (3.4) holds for $j = 0$. Then the solution of the last system will be as follows:

$$\eta^1(t) = \alpha_0(t) \in C^\infty([0,T], \mathbf{C}) \text{ is an arbitrary function,}$$

$$\eta^2(t) = \frac{(P_0(t), \chi_2(t))}{-\lambda_2(t)},$$

therefore

$$w_0(t) = \alpha_0(t)\varphi_0(t) + \frac{(P_0(t), \chi_2(t))}{-\lambda_2(t)}\varphi_2(t). \tag{3.6}$$

Since $\lambda_1(t)$ is not an eigenvalues of the matrix $A(t)$, the system (3.5_1) is uniquely solvable in the space $C^\infty([0,T], \mathbf{C^2})$:

$$\begin{aligned} w_1(t) &= [\lambda_1(t)I - A(t)]^{-1} P_1(t) \\ &\equiv \tfrac{(P_1(t), \chi_0(t))}{\lambda_1(t)}\varphi_0(t) + \tfrac{(P_1(t), \chi_2(t))}{\lambda_1(t) - \lambda_2(t)}\varphi_2(t). \end{aligned} \tag{3.7}$$

Let us turn to system (3.5_2). Making a transformation $w_2(t) = = \Phi(t)\xi \equiv \Phi(t) \cdot \{\xi^1, \xi^2\}$ in it, we get the system

$$[\lambda_2(t)I - \Lambda(t)]\,\eta = \Phi^{-1}(t)P_2(t) \Leftrightarrow \begin{cases} \lambda_2(t) \cdot \xi^1 = (P_2(t), \chi_0(t)), \\ 0 \cdot \xi^2 = (P_2(t), \chi_2(t)). \end{cases}$$

For the solvability of this system, it is necessary and sufficient that identity (3.4) holds for $j = 2$. Then the solution of the last system will be as follows:

$$\xi^2(t) = \alpha_2(t) \in C^\infty([0,T], \mathbf{C}) \text{ is an arbitrary function,}$$

$$\xi^1(t) = \frac{(P_2(t), \chi_0(t))}{\lambda_2(t)},$$

therefore

$$w_2(t) = \alpha_2(t)\varphi_2(t) + \frac{(P_2(t), \chi_0(t))}{\lambda_2(t)}. \tag{3.8}$$

Since conditions of the absence of resonance are satisfied, system (3.5_m) has a unique solution for each m $(2 \le |m| \le N_P)$:

$$w^{(m)}(t) = [(m, \lambda(t))I - A(t)]^{-1} P^{(m)}(t), \quad 2 \le |m| \le N_P. \tag{3.9}$$

Thus, for the solvability of the system (3.3) in U it is necessary and sufficient to satisfy conditions (3.4). $\square$

Remark 1. If conditions (3.4) are satisfied, then, as can be seen from (3.6) – (3.9) (taking into account $w_0(t) = \Phi(t)\eta,\ w_2(t) = = \Phi(t)\xi$, that system (3.3) has the following solution in the space

$$w(t,\tau) = \alpha_0(t)\varphi_0(t) + \tfrac{(P_0(t),\chi_2(t))}{-\lambda_2(t)}\varphi_2(t) + \left[\tfrac{(P_1(t),\chi_0(t))}{\lambda_1(t)}\varphi_0(t)\right.$$
$$\left.+\tfrac{(P_1(t),\chi_2(t))}{\lambda_1(t)-\lambda_2(t)}\varphi_2(t)\right] e^{\tau_1} + \left[\alpha_2(t)\varphi_2(t) + \tfrac{(P_2(t),\chi_0(t))}{\lambda_2(t)}\varphi_0(t)\right] e^{\tau_2} \tag{3.10}$$
$$+\sum_{|m|=2}^{N_P}\left\{\left[(m,\lambda(t))I - A(t)\right]^{-1} P^{(m)}(t)\right\} e^{(m,\tau)},$$

where $\alpha_j(t) \in C^\infty\left([0,T],\mathbf{C}\right)$ are arbitrary functions, $j = 0,2$.

We will not prove the unique solvability of iterative problems. Note that when solving two consecutive iterative problems (3.5_m) and (3.5_{m+1}), the solution to the problem (3.5_m) in the space U will be found uniquely.

4. **Construction of solutions to iterative problems**

Consider the first iterative problem and construct its solution in the space U. Since the right-hand side $P(t,\tau) = P^{(0)}(t,\tau) + P^{(1)}(t,\tau) + +P^{(2)}(t,\tau) \equiv \{h_1(t) + +h_2(t)e^{\tau_1}\sigma, 0\}$ in it that is, the orthogonality conditions (3.4) for the problem are fulfilled automatically. Therefore, the problem (3.1_0) has the following solution (see formula (3.10)):

$$w_0(t,\tau) = \alpha_0^{(0)}(t)\varphi_0(t)$$
$$+\left[\tfrac{(H(t),\chi_0(t))}{\lambda_1(t)}\varphi_0(t) + \tfrac{(H(t),\chi_2(t))}{\lambda_1(t)-\lambda_2(t)}\varphi_2(t)\right]\sigma e^{\tau_1} \tag{4.1}$$
$$+\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2},$$

where $\alpha_j^{(0)}(t) \in C^\infty\left([0,T],\mathbf{C}\right)$ for now, are arbitrary functions, $j = = 0,2$. Subjecting (4.1) to the initial condition $w_0(0,0) = w^0$, we will have

$$\alpha_0^{(0)}(0)\varphi_0(0) + \left[\tfrac{(H(0),\chi_0(0))}{\lambda_1(0)}\varphi_0(0) + \tfrac{(H(0),\chi_2(0))}{\lambda_1(0)-\lambda_2(0)}\varphi_2(0)\right]\sigma$$
$$+\alpha_2^{(0)}(0)\varphi_2(0) = w^0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \alpha_0^{(0)}(0)\varphi_0(0)+\left[\tfrac{(H(0),\chi_0(0))}{\lambda_1(0)}\varphi_0(0)\right]\sigma$$

$$+\alpha_2^{(0)}(0)\varphi_2(0)+\left[\tfrac{(H(0),\chi_2(0))}{\lambda_1(0)-\lambda_2(0)}\varphi_2(0)\right]\sigma=w^0.$$

Multiplying this equality scalarly by $\chi_s(0), s=0,2$, we have

$$\begin{aligned}\alpha_0^{(0)}(0)&=(w^0,\chi_0(0))-\left[\frac{(H(0),\chi_0(0))}{\lambda_1(0)}\right]\sigma,\\ \alpha_2^{(0)}(0)&=-\left[\frac{(H(0),\chi_2(0))}{\lambda_1(0)-\lambda_2(0)}\right]\sigma=0.\end{aligned}\tag{4.2}$$

In this case, the system (3.1_1) takes the form

$$\begin{aligned}&L_0w_1(t,\tau)=-\tfrac{\partial}{\partial t}\Big(\alpha_0^{(0)}(t)\varphi_0(t)\\ &+\left[\tfrac{(H(t),\chi_0(t))}{\lambda_1(t)}\varphi_0(t)+\tfrac{(H(t),\chi_2(t))}{\lambda_1(t)-\lambda_2(t)}\varphi_2(t)\right]\sigma e^{\tau_1}\\ &+\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2}\Big)+A_1(t)\Big(\alpha_0^{(0)}(t)\varphi_0(t)\\ &+\left[\tfrac{(H(t),\chi_0(t))}{\lambda_1(t)}\varphi_0(t)+\tfrac{(H(t),\chi_2(t))}{\lambda_1(t)-\lambda_2(t)}\varphi_2(t)\right]\sigma e^{\tau_1}+\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2}\Big)\\ &+F\Big(\alpha_0^{(0)}(t)\varphi_0(t)+\left[\tfrac{(H(t),\chi_0(t))}{\lambda_1(t)}\varphi_0(t)+\tfrac{(H(t),\chi_2(t))}{\lambda_1(t)-\lambda_2(t)}\varphi_2(t)\right]\sigma e^{\tau_1}\\ &+\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2},t\Big)\\ &+e^{\tau_2}\textstyle\int_0^t G(t,s)\alpha_2^{(0)}(s)\varphi_2(s)ds+H(t)e^{\tau_1}\sigma\equiv P^{(1)}(t,\tau).\end{aligned}\tag{4.3}$$

This system will be solvable in the space U if and only if the free term $P_0(t)$ and coefficient $P_2(t)$ at the exponential e^{τ_2} (measurement of $|m|=1$) in its right-hand side $P^{(1)}(t,\tau)$ satisfies the orthogonality conditions (3.4). Select in the right-hand side functions $P_0(t)$ and $P^{(1)}(t,\tau)$. Using the Taylor formula, we will have

$$F\left(\alpha_0^{(0)}(t)\varphi_0(t)+\left[\tfrac{(H(t),\chi_0(t))}{\lambda_1(t)}\varphi_0(t)+\tfrac{(H(t)\sigma,\chi_2(t))}{\lambda_1(t)-\lambda_2(t)}\varphi_2(t)\right]e^{\tau_1}\right.$$

$$\left.+\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2},t\right)=F\left(\alpha_0^{(0)}(t)\varphi_0(t),t\right)$$

$$+\tfrac{\partial F\left(\alpha_0^{(0)}(t),t\right)}{\partial w}\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2}+r(e^{\tau_2},t),$$

where the function $r(e^{\tau_2},t)$ contains terms with $e^{(m,\tau)}$ of dimension $|m|\geq 2$. Then we obtain

$$\hat{P}^{(1)}(t,\tau)\equiv P_0\left(t\right)+P_1\left(t\right)e^{\tau_2}$$

$$=-\left(\alpha_0^{(0)}(t)\varphi_0(t)\right)^{\bullet}+A_1(t)\left(\alpha_0^{(0)}(t)\varphi_0(t)\right)+F\left(\alpha_0^{(0)}(t),t\right)$$

$$-\left(\alpha_2^{(0)}(t)\varphi_2(t)\right)^{\bullet}e^{\tau_2}+A_1(t)\left(\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2}\right)$$

$$+\frac{\partial F\left(\alpha_0^{(0)}(t),t\right)}{\partial w}\alpha_2^{(0)}(t)\varphi_2(t)e^{\tau_2}+e^{\tau_2}\int\limits_0^t G(t,s)\alpha_2^{(0)}(s)\varphi_2(s)ds.$$

Now it is easy to write out orthogonality conditions (3.4) for problem (4.3) (take into account that $\dot{\varphi}(t)\equiv 0)$:

$$\begin{aligned}
&\dot{\alpha}_0^{(0)}(t)=(A_1(t)\varphi_0(t),\chi_0(t))\alpha_0^{(0)}(t)+F\left((\alpha_0^{(0)}(t)\varphi_0(t),\chi_0(t)),t\right),\\
&\dot{\alpha}_2^{(0)}(t)=(A_1(t)\varphi_2(t)-\dot{\varphi}_2(t),\chi_2(t))\alpha_2^{(0)}(t)\\
&+\int\limits_0^t(G(t,s)\varphi_2(s),\chi_2(s))\alpha_2^{(0)}(s)ds\\
&+\left(\tfrac{\partial F\left(\alpha_0^{(0)}(t)\varphi_0(t),t\right)}{\partial w}\varphi_1(t),\chi_2(t)\right)\alpha_2^{(0)}(t).
\end{aligned}\tag{4.4}$$

The second equation (4.4), together with the zero initial condition (4.2), is a linear integro-differential Cauchy problem with coefficients continuous on the segment $[0,T]$, therefore it has the solution $\alpha_2^{(0)}(t)\equiv 0$. For the function $\alpha_0^{(0)}\left(t\right)$ we obtain the nonlinear Cauchy problem (take into account the form $A_1(t)$ and $\varphi_0(t) = \{1, 0\}$)

$$\begin{aligned} &\dot{\alpha}_0^{(0)}(t) = -\tfrac{K(t,t)}{\mu(t)}\alpha_0^{(0)}(t) + f\left((\alpha_0^{(0)}(t)\varphi_0(t), \chi_0(t)), t\right), \\ &\alpha_0^{(0)}(0) = (w^0, \chi_0(0)) - \left[\tfrac{(H(0)\sigma, \chi_0(0))}{\lambda_1(0)}\right] \end{aligned} \tag{4.5}$$

which solvability in the whole on a segment $[0, T]$ is problematic. The study of this problem is an independent and very nontrivial problem. Therefore, we introduce one more condition:

4) *Problem (4.5) is solvable on the interval* $[0, T]$.

In this case, the solution (4.1) of the first iterative problem (3.1_0) will be found in the space U in the form

$$w_0(t, \tau) = \alpha_0^{(0)}(t)\varphi_0(t) + \left[\frac{(H(t), \chi_0(t))}{\lambda_1(t)}\varphi_0(t) + \frac{(H(t), \chi_2(t))}{\lambda_1(t) - \lambda_2(t)}\varphi_2(t)\right]\sigma e^{\tau_1}.$$

It, as mentioned above, does not contain boundary layer functions. As for the following iterative problems (3.1_k), $k \geq 1$, for them the equation for the functions $\alpha_2^{(k)}(t)$ $(k = 1, 2, 3, ...,)$ will be linear, and therefore their solvability in the whole on a segment $[0, T]$ is obvious.

Remark 2. Condition 4) will be satisfied (see [2], pp. 412–413) if we require that there exist a constant γ such that for all $(t, y) \in [0, T] \times \mathbf{R}$ the inequality

$$-\frac{K(t, t)}{\mu(t)} + \frac{\partial f(y, t)}{\partial y} \leq \gamma \tag{4.6}$$

holds. This inequality holds, for example, for the problem

$$\varepsilon\frac{dy}{dt} = \int_0^t e^{\frac{s-t}{\varepsilon}} K(t, s)y(s, \varepsilon)ds - y^3 + h(t)e^{\frac{i\beta(t)}{\varepsilon}}, \; y(0, \varepsilon) = y^0. \tag{4.7}$$

Inequality (4.6) in this case has the form

$$K(t, t) - 3y^2 \leq \gamma \qquad \Leftrightarrow \qquad -3y^2 \leq \gamma - K(t, t).$$

It is fulfilled if we take a constant $\gamma > 0$ so that $\gamma > \max_{t\in[0,T]} K(t,t)$. In this case, problem (4.5) will have the following solution

$$\alpha_0^{(0)}(t) = \frac{\sqrt{(2(\alpha_0^{(0)}(0))^2(\int\limits_0^t (e^{\int\limits_0^s K(\theta,\theta)\,d\theta})^2\,ds) + 1)e^{2\int\limits_0^t K(\theta,\theta)\,d\theta}}\,\alpha_0^{(0)}(0)}{2(\alpha_0^{(0)}(0))^2(\int\limits_0^t (e^{\int\limits_0^s K(\theta,\theta)\,d\theta})^2\,ds) + 1} \tag{4.8}$$

where $\alpha_0^{(0)}(0) = y^0 - \left[\frac{(H(0)\sigma,\chi_0(0))}{\lambda_1(0)}\right] \equiv y^0 - \frac{h(0)}{i\beta'(0)}\sigma$.

Under conditions 1) – 4), one can construct series (2.6) with coefficients $w_k(t,\tau) \in U$. As in [6, 7], we prove the following result.

THEOREM 4.1. *Let conditions 1) – 4) be satisfied for the system (2.1). Then, for $\varepsilon \in (0,\varepsilon_0](\varepsilon_0 > 0$ is sufficiently small), the system (2.1) has a unique solution $w(t,\varepsilon) \in \mathrm{C}^1([0,T],\mathbf{C^2})$ and the estimate*

$$||w(t,\varepsilon) - w_{\varepsilon N}(t)||_{C[0,T]} \le c_N\varepsilon^{N+1}, N = 0,1,2,...$$

holds; here $w_{\varepsilon N}(t)$ is the restriction (at the $\tau = \frac{\psi(t)}{\varepsilon}$) of N-th partial sum of series (2.6) (with coefficients $w_k(t,\tau) \in U$, satisfying the iterative problems (3.1_k)), and the constant $c_N > 0$ does not depend on ε at $\varepsilon \in (0,\varepsilon_0]$.

Since the solution $y(t,\varepsilon)$ of the original problem (1.1) is the first component of the vector function $w(t,\varepsilon)$, then for it (under conditions 1) – 4)), the estimate

$$||y(t,\varepsilon) - y_{\varepsilon N}(t)||_{C[0,T]} \le c_N\varepsilon^{N+1}, N = 0,1,2,...,$$

holds, where $c_N > 0$ does not depend on ε at $\varepsilon \in (0,\varepsilon_0]$.

Example. Applying the algorithm developed by us for the integro-differential problem (4.7), we obtain the following leading term of the asymptotics of the solution to this problem:

$$y_{\varepsilon 0}(t) = \frac{\sqrt{(2(\alpha_0^{(0)}(0))^2(\int\limits_0^t (e^{\int\limits_0^s K(\theta,\theta)\,d\theta})^2\,ds) + 1)e^{2\int\limits_0^t K(\theta,\theta)\,d\theta}}\,\alpha_0^{(0)}(0)}{2(\alpha_0^{(0)}(0))^2(\int\limits_0^t (e^{\int\limits_0^s K(\theta,\theta)\,d\theta})^2\,ds) + 1}$$

$$+\frac{h(t)}{i\beta'(t)}e^{\frac{i\beta(t)}{\varepsilon}},$$

where $\alpha_0^{(0)}(0) = y^0 - \frac{h(0)}{i\beta'(0)}\sigma$. Hence, it can be seen that the exact solution $y(t,\varepsilon)$ of the problem (4.7), leaving the point $y = y^0$ at the moment $t = 0$, performs fast oscillations at $\varepsilon \to +0$ about function (4.8) on a half-interval $(0,T]$, without tending to any limit. Function (4.8) is an analogue of the degenerate solution of problem (4.7). However, it is impossible to obtain it from (4.7) at $\varepsilon = 0$. This distinguishes singularly perturbed problems with a zero operator of the differential part from classical problems like

$$\varepsilon\frac{dy}{dt} = A(t)\,y + \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds + \varepsilon f(y,t)$$
$$+h(t)e^{\frac{i\beta(t)}{\varepsilon}}, \ \ y(0,\varepsilon) = y^0$$

with nonzero operator $A(t)$.

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